ELECTRIC

AND

MAGNETIC CIRCUITS

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PREFACE

This, the introductory volume of a treatise on Electrical Engineering, deals with the fundamental principles of Electricity and Magnetism, and explains fully all the essential relationships of Electric and Magnetic Circuits met with in continuous current working, and it is hoped that the subjects of generation, transmission, distribution and application of electrical energy may receive full and adequate treatment in the succeeding volumes without requiring so much introductory matter, devoted to Electrical Engineering principles, terms and definitions, as is customary in many of the standard technical treatises. Furthermore, the experience of the author leads him to the conclusion that but few engineers are able to take full advantage of the classical works with which electrical literature abounds, simply and solely because they have not mastered thoroughly the elements of the Arithmetic of Electrical Engineering. The cultivation of the power and ability to calculate readily and accurately is just as important as the cultivation of the ability to do practical work; and to emphasize this fact the opinions of two eminent electrical engineers and educational authorities may be quoted. Twenty years ago Lord Kelvin made the following statement at a meeting of the Institution of Civil Engineers: 'In physical science a first essential step in the direction of learning any subject is to find principles of numerical reckoning, and methods for practically measuring some quality connected with it. I often say that when you can measure what you are speaking about, and explain it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind.' More recently, Professor Ayrton, F.R.S., in his inaugural address delivered before the Institution of Electrical Engineers, said, 'Students of electrical engineering should be well practised in calculating and designing, and should obtain sufficient acquaintance with methods of attacking new problems, so as not to be daunted when they meet with them in after life.' As many otherwise useful and excellent text-books

on Electrical Engineering give very little or no opportunity for the student to obtain this essential acquaintance with the quantitative side of the subject, there appears to be a real demand for a book that does supply this want, and this the author claims for the present volume.

The plan adopted in this book to meet this demand has been to deduce the fundamental principles of Electrical Engineering from the system of practical electrical units, now universally used as a basis, and to deal with the various elements of the subject in the same order that the engineer meets with them in practical everyday work. An earnest endeavour has been made to introduce nothing but the simplest mathematics, for which reason some of the formulae have not been so rigidly proved as would have been possible had the higher mathematics been applied, but the author expresses the hope that the book will be educational in character and a real aid to electro-technical instruction. Whilst there is not much room for originality as regards the matter thus presented, an attempt has been made to introduce the reader to the fundamental principles of the subject without assuming any previous knowledge, and by giving a large number of worked-out examples to offer a considerable amount of useful information to the engineer and student alike. At the same time a good deal is left for the student to do for himself, and except in the earlier portions of the book the method of obtaining the various relationships is merely indicated in some cases, and the simplification of the formulae is left as an exercise for the student. If he supplies the connecting link from what has been given previously he will derive much benefit; if not, his progress will be anything but thorough. The exercises to be worked—nearly 700 in number, and which are a special feature of the book—were primarily prepared for use in the author's classes in Electrical Engineering at the University College, Sheffield, and it is hoped that their practical character may appeal to the practical man as well as to the engineering student. A very large proportion of the exercises are original, and the others have been taken from the examination papers of the Board of Education and the City and Guilds of London Institute, and it will be found that the special needs of the engineering student have been considered rather than those of the student of pure Physics. To the teacher the collection

of exercises is offered as a means of saving the time now devoted to the dictation of exercises and homework. To the student, especially if he be taking the subject privately, the exercises should be equally useful.

As regards the descriptive portion of the book it is believed that this volume contains all the essential and elementary matter relating to continuous current circuits, and whilst a considerable portion deals with the practical application of electricity it is not intended to be a text-book on the design of distributing mains or continuous current machinery, but to indicate how the fundamental principles may be applied to the many and varied problems encountered in practical work. The subject of alternating and polyphase current circuits is not dealt with at all, as this branch of Electrical Engineering is to receive special treatment in a second volume, which is in an advanced stage of preparation.

In the preparation of this work the author has received much assistance from the classical works and lectures of prominent electrical engineers and the technical press, and desires to acknowledge his indebtedness to these authorities for the many ideas which have enabled him to prepare the following chapters. He has also to express his thanks to His Majesty's Stationery Office for permission to use the table of logarithms and anti-logarithms given in the appendix.

As it is almost impossible in the first edition of a book of this nature to detect every slip which may occur during its preparation the author requests his readers who may detect any error or ambiguity to kindly notify him of the same so as to make the book as reliable as possible.

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ELECTRIC AND MAGNETIC CIRCUITS

CHAPTER I

PRACTICAL ELECTRICAL UNITS

Section I. The Electric Current and its Unit. The Ampere.

§ 1. Introduction. The subject of electric currents and the transference of electricity from one point to another forms the most suitable introduction to the study of electric circuits, since the electric current is that which is utilized in all the practical applications of electricity. Furthermore, the units of current and quantity of electricity may be defined without reference to any other electrical unit. In practice it is convenient and usual to compare the passage of electricity along a conductor to the flow of an incompressible fluid in a pipe, but we must point out, once for all, that electricity is not materialistic in its nature, and that the comparison is purely and simply arbitrary. As a matter of fact the innate nature of electricity is beyond our present knowledge. We know how to produce it, and also how to use it; and it is common knowledge that certain effects are always produced and always accompany the transference of electricity from one point to another. In electric lighting, for instance, the heating effect of electricity is utilized to produce light, and, speaking generally, we may state that the existence of a current of electricity can only be recognized by the effects which it produces. It is also remarkable that these effects may be concentrated to any degree and wherever they may be required, and it is in consequence of this that electrical energy is of such immense practical value.

Electricity may be produced in a variety of ways, i. e. by friction, by chemical and by mechanical means, and in each case the production of electricity can only be effected at the expense of some form of energy, and as the result the material electrified is put into

a definite physical condition, and is capable of doing work; in other words, electrification is accompanied by the production of energy. Electricity produced by means of batteries (chemical method) is suitable for experimental and signalling purposes only. whilst for commercial undertakings on a large scale electricity is produced by mechanical means, and for lighting, power and traction purposes we have to depend upon the dynamo as the generator of electricity.

§ 2. **Electric Currents and the Ampere.** Commercially, we are mainly concerned with electricity in motion, and in ordinary parlance we speak as if electricity were passing or flowing along a conductor, just as water flows along a pipe, and it is from this arbitrary notion that we get the idea of currents of electricity. When we speak of a wire carrying electricity it is merely a statement of the fact that the wire is in a different physical state to what it is when electricity does not pass. This idea that electricity flows along a wire has been sanctioned by custom and is very convenient, although, as a matter of fact, we do not know whether there is any motion at all, whether positive or negative electricity flows or with what velocity it flows. According to modern views this flow of electricity refers to a transference of energy in the form of electro-magnetic wave motion about the wire as a centre, and the wire and the space surrounding it are energized.

When we remember that the word 'current' means simply 'rate of flow,' it will be clear that the term 'current of electricity' simply means 'rate of flow of electricity.' And just as we can measure the rate of flow of water so we can measure currents of electricity, and in definite units, too, great care having been taken to arrange a comprehensive system of electrical units and standards, a complete system of which was legalized by the Board of Trade in 1894. The unit quantity of water is the gallon, cubic inch or cubic foot, whilst the practical unit of quantity of electricity is termed the *coulomb*, after a celebrated French physicist, Coulomb. The unit rate of flow of water is the gallon per second, cubic inch or cubic foot per second as the case may be; or, in other words, is that quantity of water in gallons, cubic inches or cubic feet which passes any section of a pipe in one second. Similarly, the practical unit of a current of electricity is that quantity of electricity in coulombs which passes any section of a conductor in one second; in other words, unit current or unit rate of flow of electricity is the coulomb per second and is termed the *ampere*, after the well-known French physicist, Ampère.

The average strength of a current is therefore measured by the time rate of flow of electricity across the section of the conductor,

and if we let C denote the strength of the current, Q the quantity of electricity, then in the language of the calculus, we have

$$C = \frac{\delta Q}{\delta t}$$

and the *ampere* may be defined as *the time rate of change of the coulomb per second.*

Electric currents may be classified according to their characteristic properties into three varieties, (1) *continuous currents,* (2) *pulsatory currents,* and (3) *alternating currents.* Continuous currents are those which have constant direction and magnitude, and if not of constant magnitude do not vary in a periodic manner. Pulsatory currents are those which maintain the same direction of flow, but whose strength varies in a periodic manner. Alternating currents are those which fluctuate in direction and magnitude in a periodic manner. In this book we shall only consider continuous currents. When the magnitude of the current does not vary with the time we have a *steady* current and a constant current circuit in which there is no accumulation of electricity at any point, and the number of coulombs passing across any section of a simple circuit is the same at every point of the circuit.

Since electricity cannot be seen and treated as a material substance, advantage is taken of the effects or phenomena which always accompany the transference of electricity as means to measure and compare currents of electricity. The effects which may be utilized for this purpose are (1) the magnetic effects, (2) the heating effects, and, (3) the chemical effects. In other words, currents of electricity (1) set up magnetic fields, magnetize pieces of iron and steel, and deflect magnetic needles ; (2) raise the temperature of the conductors through which they pass, and (3) decompose compound liquids and bring from them their constituent elements. Measurements and comparisons of currents made by methods depending upon these effects give concordant results, but it is advisable in the case of standard measurements to utilize only such properties which give results directly proportional to the current, and experience proves that in no other case is the effect so marked, and the determination so exact and easy, as in the chemical or electrolytic method—that is by electrolysis. Our knowledge of electro-chemistry dates from the time of Faraday, who was the first to announce the exact laws of electrolysis in his 'Experimental Researches.'

The subject of electro-chemistry will be treated fully in a later chapter, and for the present we shall only deal with the electro-deposition of metals in a brief manner, so as to explain the method of measuring and comparing currents.

When an *electrolyte*, i. e. a compound liquid with a metallic salt in solution, is interposed in a circuit, so that electricity may be passed through it by means of plates or electrodes immersed in the liquid, it is found that the electrolyte is decomposed, and the separated elements pass to the electrodes, and in the case where metals are set free they enter into combination with an electrode. This decomposition is termed electrolysis, and the products of the decomposition are termed *ions*. Thus, if copper sulphate forms the electrolyte and copper plates be used as the electrodes, the passage of electricity through the electrolyte is accompanied by a deposit of copper on one electrode, whilst copper is dissolved from the other plate, by which means the strength of the solution is maintained fairly constant. One of the electrodes is thus a *gain* plate, and the other a *loss* plate. Technically the gain plate is known as the *cathode*, and this is the electrode which leads the electricity *out of* the solution, and the loss plate is termed the *anode* (electrode leading the electricity *into* the solution). Upon weighing the gain plate, at the beginning and end of the test, the weight of copper deposited may be accurately determined. It is found that the amount of chemical decomposition which takes place in unit time is a definite and fixed quantity for unit current, and is the same at all points of a simple circuit. From a large number of experiments it has been found that unit current deposits 0·0003281 of a gramme of copper per second from a solution of copper sulphate, and deposits 0·001118 of a gramme of silver per second from a solution of silver nitrate. It is also found that the quantity of metal deposited is directly proportional to the current.

These quantities—0·0003281 for copper, and 0·001118 for silver—are called the *electro-chemical-equivalents* of copper and silver respectively, and similarly the weight in grammes of any element deposited per second by unit current is the electro-chemical-equivalent of that element. The following are the electro-chemical-equivalents of the most common elements, a more complete table being given on page 122.

Hydrogen . .	0·000010384.
Silver . . .	0·001118.
Copper . . .	0·0003281.
Nickel . . .	0·0003043.
Zinc	0·00033698.

Electrolysis has long been acknowledged as a reliable basis for the practical determination of the magnitudes of currents and for the calibration of ammeters, and for this reason the Board of Trade adopted the results of electro-chemical decomposition as the basis for the legal definition of the ampere, the unit of current. We may

mention here that the Board of Trade recommend a solution of silver nitrate for the determination of the ampere, and the following is the Board of Trade specification for the determination :—

SPECIFICATION A.

In the following specification the term silver voltameter means the arrangement of apparatus by means of which an electric current is passed through a solution of nitrate of silver in water. The silver voltameter measures the total electrical quantity which has passed during the time of the experiment, and by noting this time the time average of the current, or if the current has been kept constant the current itself, can be deduced.

In employing the silver voltameter to measure currents of about one ampere the following arrangements should be adopted : The cathode on which the silver is to be deposited should take the form of a platinum bowl not less than 10 centimetres in diameter, and from four to five centimetres in depth.

The anode should be a plate of pure silver some 30 square centimetres in area and two or three millimetres in thickness.

This is supported horizontally in the liquid near the top of the solution by a platinum wire passed through holes in the plate at opposite corners. To prevent the disintegrated silver which is formed on the anode from falling on to the cathode, the anode should be wrapped round with pure filter paper, secured at the back with sealing-wax.

The liquid should consist of a neutral solution of pure silver nitrate, containing about 15 parts by weight of the nitrate to 85 parts of water.

The resistance of the voltameter changes somewhat as the current passes. To prevent these changes having too great an effect on the current, some resistance besides that of the voltameter should be inserted in the circuit. The total metallic resistance of the circuit should not be less than 10 ohms.

Method of making a Measurement.

The platinum bowl is washed with nitric acid and distilled water, dried by heat, and then left to cool in a desiccator. When thoroughly dry it is weighed carefully.

It is nearly filled with the solution, and connected to the rest of the circuit by being placed on a clean copper support to which a binding screw is attached. This copper support must be insulated.

The anode is then immersed in the solution so as to be well covered by it, and supported in that position ; the connexions to the rest of the circuit are made.

Contact is made at the key, noting the time of contact. The current is allowed to pass for not less than half an hour, and the time at which contact is broken is observed. Care must be taken that the clock used is keeping correct time during this interval.

The solution is now removed from the bowl, and the deposit is washed with distilled water and left to soak for at least six hours. It is then rinsed successively with distilled water and absolute alcohol, and dried in a hot-air bath at a temperature of about 160 deg. C. After cooling in a desiccator it is weighed again. The gain in weight gives the silver deposited.

To find the current in amperes, this weight, expressed in grammes, must be divided by the number of seconds during which the current has been passed, and by 0·001118.

The result will be the time average of the current, if during the interval the current has varied.

In determining by this method the constant of an instrument the current should be kept as nearly constant as possible, and the readings of the instrument observed at frequent intervals of time. These observations give a curve from which the reading corresponding to the mean current (time average of the current) can be found. The current, as calculated by the voltameter, corresponds to this reading.

From the above specification it is evident that every precaution must be taken to ensure accurate results in the standardization tests, and, although it is clear from the table of electro-chemical-equivalents, already given, that silver nitrate is the best electrolyte to employ for measuring currents, since its electro-chemical-equivalent is the greatest, still silver is somewhat difficult to manage. For instance, if the exact current is not passed for the size of the plates used, it is difficult to get a firm deposit; and the silver will be crystallized and non-adherent, causing a slight loss when finally washing the cathode. Students, however, may obtain very reliable results with the copper voltameter, if care be taken and the following precautions, in addition to those mentioned in the above specification, be carried out.

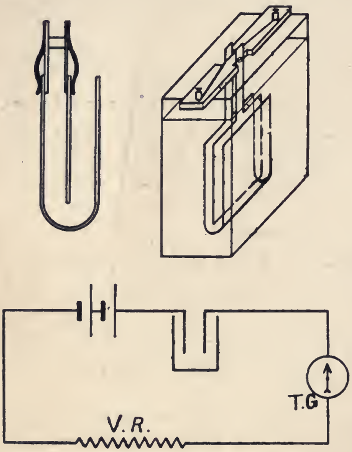


Fig. 1.

The copper sulphate solution should have a density of 1·15 to 1·18 and be made of pure copper sulphate crystals dissolved in distilled water, after which about one per cent. of strong sulphuric acid should be added. The arrangement for supporting the copper plates is shown in Fig. 1, and it is advisable that the middle plate—the cathode—should have an area of four or five sq. inches per ampere, so that from eight to ten sq. inches of plate (cathode) surface may be exposed per ampere. The plates should be about $\frac{1}{2}$ an inch apart, the two outside plates being arranged so as to form the anode. By this arrangement copper is deposited on both sides of the cathode. A tangent galvanometer, T.G, or other measuring instrument, should be connected in the circuit, so that the constancy of the current may be maintained throughout the test by means of the variable resistance, V.R. Thirty minutes is a suitable time for the duration of each test. It is also useful to check the results of a test by inserting in the

circuit a suitable ampere-meter or ammeter which is, as is well known, a direct reading instrument so arranged that usually the magnetic effect of a current indicates the current in amperes by means of a pointer on a dial, thus doing away with the necessity of weighing &c. It may also be pointed out that the method of determining the strength of a current by electrolysis is the method of calibrating an ammeter.

§ 3. **Board of Trade Definition of the Ampere.** *The Ampere is the name of unit current, and is that unvarying electric current which, when passed through a solution of nitrate of silver in water, in accordance with the specification given above, deposits silver at the rate of* 0·001118 *of a gramme per second.* In a similar manner we may define the coulomb, the unit of quantity of electricity; thus, the coulomb is the unit of quantity of electricity, and is that definite quantity which, when passed through a solution of nitrate of silver in water, deposits 0·001118 of a gramme of silver.

It is therefore clear that the ampere is a compound unit, and includes the element time; also that a current of electricity of one ampere does a definite amount of work per second, and thus conveys the idea of '*rate*' *of transfer* and *time-rate of change.* As an example, we may mention that many glow-lamps (120 volts) give 16 candle power of light when supplied with electricity at the rate of half a coulomb per second, that is, with half an ampere. We may thus write

$$\text{current in amperes} = \frac{\text{quantity of electricity in coulombs}}{\text{time in seconds}}$$

and quantity of electricity in coulombs $= \left\{ \begin{array}{c} \text{current} \\ \text{in amperes} \end{array} \right\} \times \left\{ \begin{array}{c} \text{time} \\ \text{in seconds} \end{array} \right\}$

or, in symbols

$$\left. \begin{array}{l} C = \dfrac{Q}{t} \\ Q = C.t. \end{array} \right\} \quad \text{and} \quad . \quad . \quad . \quad . \quad (1)$$

§ 4. **Laws of Electrolysis and quantitative relationships.** We may collect the fundamental facts and relationships respecting the electrolytic determination of current strength, and explain how to obtain the quantity required. The principles of electrolysis are usually expressed in the form of two laws, known as Faraday's Laws, as follows:—

I. The amount of each electrolyte decomposed by a unit quantity of electricity is a definite and constant quantity, which is termed its electro-chemical-equivalent.

II. The amount of any electrolyte decomposed by electricity is directly proportional to the total quantity, or number of coulombs

of electricity which pass through it; in other words, it is proportional to the strength of the current and to its duration in seconds.

If e denotes the electro-chemical-equivalent of a metal, then from the first law we know that one coulomb deposits e grammes of that metal, since e is given in grammes, therefore the total weight W, in grammes, of a metal deposited by electricity is proportional to e, or

$$W \propto e \qquad I$$

by the second law

$$W \propto Q \qquad II$$

where Q = total number of coulombs used, but

$$Q = C \times t$$

and combining equations I and II we get

$$\left.\begin{aligned} W &= Q \times e \\ &= C \times e \times t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) gives the relationship between the weight of a metal of electro-chemical-equivalent e, deposited by a current of C amperes in the time t seconds. This equation may also be written

$$C \text{ (amperes)} = \frac{W}{e \times t} \quad . \quad . \quad . \quad . \quad (3)$$

which is a convenient form for calculating the current when the weight of a metal deposited and the time in seconds are known.

For practical purposes, especially for stating the capacity of storage cells or accumulators, the coulomb or ampere-second is too small a unit of quantity and is useless commercially; consequently a commercial unit equal in magnitude to 3600 coulombs is used in practice. Since the coulomb is an ampere-second and there are 3600 seconds in one hour, 3600 coulombs, i. e. 3600 ampere-seconds, form the *ampere-hour, the commercial unit of quantity. An ampere-hour is defined as that quantity of electricity which passes through a circuit, when a steady current of one ampere flows continuously for one hour, and is equivalent to 3600 coulombs.*

If Q_H denotes ampere-hours, and Q denotes coulombs, then

$$Q_H = \frac{Q}{3600}$$

and formula (2), giving the weight of metal deposited, becomes

$$W = 3600\ Q_H \times e \quad . \quad . \quad . \quad . \quad (4)$$

As is well known, some metal in the form of a cylindrical wire is used for conducting electricity between two points, and the size of cable should correspond to the intensity of the current to be carried. Many Fire Insurance Offices specify that the cross-sectional area of the conductors employed shall correspond to a certain amperage (number of amperes) per sq. inch of cross-section. This intensity of current per sq. inch is termed the *current density,* and a common value of the

current-density sanctioned by the fire offices for small conductors in this country is 1000 amperes per sq. inch. The carrying capacity will be considered in detail in chapter III, but we may point out that the current density mentioned above is liberal as regards the elevation of temperature for ordinary currents, but that it is not a sound or safe basis for large currents, since it is not based upon uniform elevations of temperatures with conductors of different sizes. Clearly the current allowed by a given current density increases as the square of the diameter as the conductors increase in size, whilst the radiating surface, which is the determining factor respecting elevations of temperature, only increases as the diameter. The elevation of temperature consequently increases rapidly as the diameters are increased and the currents correspond to a fixed current density, and the 1000 amperes per sq. inch basis should not be followed except for small conductors.

In practice it is often necessary to determine the current which will melt fusible pieces of metal, used to protect circuits from injury when abnormal currents traverse the circuits, and Sir W. H. Preece, F.R.S., has given the following formula, connecting the fusing current with the diameter of different wires. If C is the fusing current in amperes, d = diameter of the wire in inches, and a = a constant depending upon the material used, then

$$C = a\, d^{\frac{3}{2}} \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

Preece gives the following values of a for the following metals used as fuses:—

Copper	10244
Tin	1642
Lead	1379
Alloy of tin and lead (1 : 2) .	1318

Worked Examples. The following are typical examples respecting the relationships existing between the units, the coulomb, ampere, and ampere-hour, and of problems relating to the determination of currents.

(1) How much copper will 25 amperes deposit in 25 minutes?

$$\text{Since } W = c \times e \times t$$
$$W = 25 \times 0{\cdot}0003281 \times (25 \times 60)$$
$$= 12{\cdot}30375 \text{ grammes.}$$

(2) What current will be required to deposit 1 gramme of zinc in 49 minutes?

$$C = \frac{W}{e \times t}$$
$$= \frac{1}{0{\cdot}00033698 \times 49 \times 60}$$
$$= \frac{1}{0{\cdot}9907212} = 1 \text{ ampere (approx.).}$$

(3) It is found that 100.62 grammes of silver are deposited in one hour; determine the quantity of electricity passed, and also the current in amperes.

$$\text{Since } W = Q \times e$$

$$Q = \frac{W}{e} = \frac{100.62}{0.001118}$$

$$= 90000 \text{ coulombs,}$$

$$\text{and } Q = C \times t$$

$$\therefore C = \frac{Q}{t} = \frac{W}{e \times t}$$

$$= \frac{90000}{60 \times 60} = 25 \text{ amperes.}$$

(4) Determine the number of pounds of copper deposited by 1000 ampere-hours of electricity.

Note.—In electro-chemical works it is useful, and often necessary, to prepare tables which give the number of pounds or grammes of the different metals prepared electrolytically per ampere-hour, or the number of ampere-hours required to deposit 1000 grammes or pounds of the metal deposited. This example is inserted to show the method of preparing such tables.

$$\text{Since } W = 3600 \times Q_H \times e$$

$$\therefore \text{ weight of copper} = 3600 \times 1000 \times 0.0003281$$

$$= 1181.16 \text{ grammes}$$

$$\text{but one pound} = 453.6 \text{ grammes}$$

$$\therefore W \text{ (lbs)} = \frac{1181.16}{453.6}$$

$$= 2.6 \text{ lbs.}$$

(5) A battery of accumulators supplies 40 amperes for 20 hours for a certain experiment. If the capacity of the battery be taken as 1000 ampere-hours, how many coulombs of electricity are not discharged?

$$40 \text{ amperes for 20 hours} = 800 \text{ ampere-hours}$$

$$\therefore \text{ quantity not discharged} = 1000 - 800$$

$$= 200 \text{ ampere-hours}$$

$$\text{and 200 ampere-hours} = 200 \times 3600$$

$$= 720000 \text{ coulombs.}$$

(6) It is found that a current of 48 amperes melts a copper fuse 0.028 inch in diameter; determine the current which will melt a copper fuse 0.02 inch in diameter.

$$\text{Since } C = a \times d^{\frac{3}{2}}$$

$$48 = a \times (0.028)^{\frac{3}{2}}$$

$$= a \times 0.004685$$

$$\therefore a = \frac{48}{0.004685} = 10245$$

therefore the fusing current to melt a copper fuse 0.02 inch in diameter is:

$$C = 10245 \times (0.02)^{\frac{3}{2}}$$

$$= 10245 \times 0.002828$$

$$= 29 \text{ amperes.}$$

EXERCISES IA.

Electric Currents.

(1) If a constant current of 50 amperes flows for 20 minutes, determine the total quantity of electricity which passes.

(2) Three thousand coulombs deposit a certain quantity of silver in 1 hour 6 minutes 40 seconds, determine the strength of the current indicated by an ammeter placed in the circuit.

(3) How many ampere-hours are equivalent to a current of 75 amperes flowing for 20 minutes?

(4) In a certain circuit the quantity of electricity which passes in 45 minutes amounts to 30 ampere-hours; determine the current which is constant.

(5) Determine how many coulombs of electricity pass through a 16 c. p. lamp, requiring 0·6 ampere and burning 12 hours.

(6) An electric light installation consists of 150 glow-lamps, each taking 0·6 ampere. Determine the total number of ampere-hours of electricity used, if 150 lamps are in use for $1\frac{1}{2}$ hours, then 120 lamps for 1 hour, and afterwards 100 lamps for 45 minutes.

(7) What weight of copper will 15 amperes deposit per hour?

(8) How many coulombs of electricity will be required to deposit 0·54774 grammes of nickel?

(9) What is the strength in amperes of a current passed through a silver voltameter for $1\frac{1}{2}$ hours, which deposits 90·558 grammes of silver?

(10) How many amperes will be required to deposit one pound of copper in 30 hours? 1 lb = 453·6 grammes.

(11) Determine the number of ampere-hours required to deposit one pound of silver.

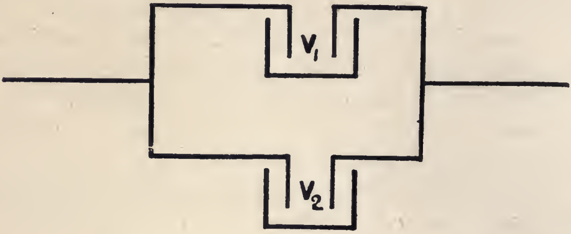
(12) How long will it take 0·1016 ampere to deposit two milligrammes of copper?

(13) Determine the number of kilogrammes of copper deposited per 1000 ampere-hours.

(14) If an electric current in one hour will deposit one ounce of silver, how much copper will a current of the same strength deposit in 10 hours? (C. and G.)

(15) A constant current from a battery of Daniell cells was passed through two voltameters placed in the same circuit, one containing nitrate of silver and the other acidulated water. Determine the weight of silver deposited and the quantity of hydrogen evolved in 15 minutes, the current strength being 1·25 amperes.

(16) In the figure V_1 is a silver voltameter, and V_2 a copper voltameter, and it is found that by suitably arranging the currents flowing in the two branches that equal quantities of silver and copper are deposited in the same time. Determine the ratio of the currents flowing in the two branches.



(17) How much more copper will 60 ampere-hours of electricity deposit than a current of 25 amperes will deposit nickel in $2\frac{1}{2}$ hours?

(18) Draw a curve, by plotting distances on a horizontal line to represent amperes, and corresponding ordinates to represent the quantities of silver deposited, and show the errors in the readings of an ammeter placed in a circuit arranged so that the ammeter may be calibrated, and in which the following quantities of silver were deposited when the ammeter registered 1, 2, 3 . . . . 10 amperes, the test in each case lasting 15 minutes; 1·0063118, 2·0125118, 3·0186559, 4·0248, 5·031, 6·0371441, 7·0433441, 8·0494882, 9·0556882, 10·0618882.

(19) Draw up a table of errors in the scale readings in the above example, marking the excess in the readings as +, and the deficit as −.

(20) A 37/16 cable has a cross-sectional area of 0·1176 sq. inch, what current may be transmitted at a current density of 750 amperes per square inch, and also of 1000 amperes per square inch?

(21) What is the current density per sq. cm. on the basis of 645 amperes per sq. inch, and what should be the area of cross-section in sq. mms. of a conductor to carry 10 amperes on this basis?

(22) It is found that a certain current melts a fuse wire of tin 0·028 inch in diameter; determine the current if the constant, *a*, for tin is 1642.

Section II. Conductors and Resistance.

§ 5. The Path of a Current. The idea of electricity being in motion implies some conducting channel, and the whole path traversed by electricity is said to form an 'electric circuit.' Electric currents are, therefore, circuital quantities, and by that we mean that electricity

only traverses complete circuits and leaves a battery or source at the same rate that it returns to it. A circuit may comprise one or more paths, and also include leakage paths, and it is the work of the engineer to keep the leakage as small as possible. As we have already inferred, some metal in the form of a wire is generally the medium by means of which electricity is transferred from one point to another, and it is an essential condition of good working and a distinct advantage when an electric circuit is formed by a material or materials which allow electricity to pass easily. Materials which do this are called conductors, and since different metals of the same form and dimensions vary greatly in their conducting powers, it is very important to know the degree in which conductors differ in this respect. This statement implies 'obstruction' to a greater or less extent to the passage of electricity, and it is because of this obstruction that the magnitude of currents to a certain extent is limited. This property of conductors, obstructing and limiting the quantity of electricity passing through a circuit, is analogous to friction, and is termed electrical resistance. Electrical resistance, then, being of the nature of friction, determines the amount of dissipation of energy accompanying the passage of electricity along a wire, and it must always be borne in mind that resistance is closely allied with a continuous waste and expenditure of energy.

Resistance, then, may be defined as *the obstruction which electricity encounters when traversing a conductor.* A good conductor offers little resistance, whilst a bad conductor offers so much resistance that it is often used to confine the electricity along a definite channel; it is then termed an 'insulator.' A perfect and ideal conductor would have no resistance, and then there would be no waste of energy; all substances, however, conduct electricity more or less, and whilst there is no substance which offers no resistance, there is no material known which obstructs the passage of electricity absolutely, nor is there any clear dividing line between conductors and insulators. Solid, liquid and gaseous matter interpose resistance, and since the magnitude of a current of electricity is limited under certain conditions by the obstruction offered by the conductors, it is obvious that the resistance of conductors plays a most important part in tho distribution of electricity for commercial purposes, and in practice we often require to compare and measure the resistance of conductors with great accuracy; consequently, a standard is required, and a suitable conductor of convenient form, given dimensions, and definite physical condition has been chosen, the electrical resistance of which is the unit of electrical resistance known as the *ohm*.

In electric lighting, power and traction work electrical resistance

is an important factor; the circuits are of great length, and although the resistance can be kept small by increasing the area of cross-section as the length increases, the question of economy, weight and first cost imposes a limit beyond which the area of cross-section must not increase. Then again, in traction work, resistance is interposed at the rubbing surfaces (trolley-wheel and at the rails), which varies considerably with circumstances; resistance at contacts are also numerous, and at the joints of the rails much resistance is also introduced, and for regulation and control resistance coils are inserted. As is well known, the resistance at the rail joints is particularly important, and every effort to keep it at a minimum is made by bonding, &c., since efficient bonding prevents leakage and waste.

In practice there is always a small current which finds its way from the tram rails back to the power-station by means of the earth, and this is rendered possible when the ground is moist, as is invariably the case below a certain depth, even when the roadway is perfectly dry. Earthy matter when dry has high resistivity, but in the case of damp ground various salts are in solution, rendering the resistivity so small that by using large earth plates buried in moist earth at the station the earth forms a good earth return with very little electrical resistance.

§ 6. The Ohm, the unit of Resistance. The Order in Council regarding units and standards of Electrical Measurements issued in 1894, specifies that the unit of electrical resistance shall be that resistance offered to an unvarying electrical current by a column of pure mercury, 106·3 centimetres long and of a constant cross-sectional area of one square millimetre, at the temperature of melting ice. Such is the standard of resistance, or *Standard Ohm*, named in honour of the physicist Dr. G. S. Ohm. Since the weight of a column of mercury can be more accurately determined than the bore of a tube one square millimetre in area, the Order in Council specifies that the weight of a column of pure mercury 106·3 centimetres in length, which offers unit resistance, should be 14·4521 grammes. The standard ohm is the legal standard of resistance, being the resistance between the terminals of the instrument, marked 'Board of Trade Ohm Standard Verified 1894,' to the passage of an unvarying electrical current, when the coil of insulated wire forming part of the aforesaid instrument and connected to the aforesaid terminals is in all parts at a temperature of 15°·4 C. The coil and instrument are deposited at the Board of Trade Standardizing Laboratory, 8, Richmond Terrace, Whitehall, London, and are preserved for reference in the same way as the bronze bar, deposited at the office of the Exchequer in London,

on which the distance between two transverse lines is, at the temperature of 62° F., the yard, the standard of length.

Very often a symbol—the small Greek letter omega, ω—is used as an abbreviation for the term ohm. Thus 10 ω signifies 10 ohms.

The factors which individually and independently affect the resistance of a conductor are the dimensions, the material, the temperature and physical condition generally. Experiment has proved that the laws which give the relation between the resistance of a conductor and the above factors are analogous to those known to exist in hydraulic circuits and to regulate the flow of a fluid in a pipe.

§ 7. **Resistance a function of the dimensions.** When a pipe is connected to a cistern of water, the level of which is assumed to be maintained constant, we know that the longer the pipe is, the greater will be the resistance to the flow, and consequently the smaller the quantity flowing per second through it, whilst every increase in the diameter of the pipe permits an increase in the rate of flow, other conditions remaining the same; thus resistance to the flow of water decreases with an increase in the size of the pipe, and increases as the length increases. In other words, the resistance of a pipe to the flow of water depends upon the dimensions of the pipe. Similarly, the resistance of a conductor to the passage of electricity is directly proportional to the length. Increased length gives increased resistance. The resistance also varies inversely as the area of the cross-section of the conductor. Doubling the area of a conductor allows twice the number of amperes to pass, other things remaining the same; in other words, the resistance is halved. These facts are comprised in the statement that resistance is a function of the dimensions of a conductor.

These particulars may be stated algebraically as follows. Let l denote the length, A the area of cross-section, and R the resistance of a conductor, then

$$R \propto l \quad \text{(resistance varies as the length).}$$

$$R \propto \frac{1}{A} \quad \text{(resistance varies inversely as the area).}$$

Combining these expressions we get

$$R \propto \frac{l}{A} \qquad \qquad (6)$$

Example I. The resistance of a certain size of copper wire has a resistance of 10 ohms per mile, therefore the following lengths of the same wire 0·05, 0·1, 5 and 100 miles will have resistances of 0·5, 1, 50 and 1000 ω respectively.

Example II. A copper wire 0·001 of a sq. inch in area allows 1·02 amperes to pass under certain conditions. What current will pass through a wire of

the same length and material, 0·004 of a sq. inch in area, under the same conditions? The conditions being the same the current passing depends only upon the resistance, and the resistance is inversely proportional to the area :

$\therefore$ the resistance of the second wire is $\frac{1}{4}$ that of the first,

$\therefore$ 4 times the current will pass, that is

current $= 1{\cdot}02 \times 4 = 4{\cdot}08$ amperes.

§ 8. **Resistance depends upon the material of which the Conductor is made.** Again, with a pipe of given dimensions the quantity of water flowing through it, under given conditions, also depends upon the material and the nature of the inside surface of the pipe, and in a similar way electrical resistance varies with different materials and their physical conditions; in other words, resistance is a physical property of the conductor itself. Now, whenever a physical property varies in different materials it is usual to make comparisons between these substances under standard conditions for convenience of reference. Thus, since the physical property, electrical resistance, varies with different materials, the resistance of a mass of known volume of the various materials in a definite form is accurately determined. Suppose a centimetre cube—a cube each side of which is one centimetre long—is interposed in an electric circuit and the resistance offered to the passage of electricity between two opposite faces of the cube at a temperature of 0° C. is found in terms of the ohm. This resistance when exactly determined is of the nature of a constant, and is known as the *Specific Resistance* or *Resistivity* of the material. As the specific resistances are very small quantities it is more convenient to express the specific resistance of a conductor in *microhms* (millionths of an ohm). Thus the specific resistance of a sample of nearly pure copper has been found to be 0·000001561 of an ohm, or 1·561 microhms.

Resistance then depends upon the material of which the conductor is made, and if σ be used to denote the specific resistance of a substance, then

$$R \propto \sigma \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Example. The specific resistance of iron is 9·065 microhms, and that of pure copper is 1·561 microhms, therefore the resistance of one mile of iron wire of a certain diameter will have $\frac{9065}{1561}$ or 5·8 times the resistance of a mile of copper wire of the same diameter. That is, copper allows nearly six times the current to flow that would flow through an iron wire of the same dimensions, and under similar conditions.

From this example it will be seen that by selecting a certain substance as a standard of reference, and comparing the specific resistances of different substances with the specific resistance of the standard, we are able to arrange a table of different materials used

as conductors, see column 5, table A, page 354, showing at a glance the relative resistances of different metals. Silver on account of its having the least specific resistance is chosen as the standard, and the constant or ratio got by comparing the specific resistance of a substance with that of the standard substance silver is known as the *relative specific resistance*, which may be defined as 'its relative resistance as compared with the resistance of the standard material of similar dimensions and under exactly similar conditions.'

Example. The resistance of a centimetre cube of silver is 1·468 microhms whilst the resistance of a similar cube of platinum is 10·917 microhms. These numbers are the specific resistances of silver and platinum, and the ratio $\frac{10917}{1468}$ or 7·436 is the relative specific resistance of platinum as compared with silver, that of silver being taken as unity.

Again, the resistance of a conductor also depends upon the purity of the material, the degree of hardness or softness, &c. When the material of a conductor is an alloy, its resistivity is always higher than that of the individual metals forming the alloy. For instance, german-silver is made of copper, zinc and nickel, the specific resistances of which are respectively 1·561, 5·751, and 12·323 microhms, whilst that of german-silver itself is 21 microhms. It will not be out of place here to point out that the magnitude of the current itself does not exert any influence upon the resistance of the conductor through which the electricity is passing, except in so far as it changes the physical state of the wire, as for instance, by raising the temperature of the wire.

It is an experimental fact that the resistance of metals increases with increase of temperature, but the exact relationship existing between variation of resistance with changes of temperature does not appear to have been determined absolutely, although much valuable experimental work in this direction has been done. Dr. Matthiessen gave the empirical formula:

$$R_t = R_0 \{1 + ct + \beta t^2\}$$

where R_t is the resistance at t° C., R_0 that at 0° C., and a, β are numerical constants. More recent determinations by Messrs. Dewar and Fleming, Swan and Rhoden, and Kennelly and Fessenden concur in giving a simple linear relation between resistance and temperature in the case of copper, and the formula

$$R_t = R_0 (1 + at)$$

is quite accurate enough for ordinary purposes. The constant a is known as the temperature coefficient, and may be taken as 0·004 per degree Centigrade for pure metals, except iron and tin. For these metals a may be taken as 0·005 and 0·007 respectively. Carbon is

an exception to the general law and has a negative temperature coefficient; the percentage variation of resistance per degree Centigrade being, according to Ayrton, $-0{\cdot}03$, i. e. $a = -0{\cdot}0003$.

§ 9. **Sizes of Conductors.** Engineers in dealing with areas usually express them in square inches, square centimetres, &c., and as is well known, the area of the cross-section of a wire—conductors being generally wires and cylindrical in form—is found by multiplying the square of the radius by 3·1416, since

$$\text{area of circle} = \pi \times r^2$$

$\pi = 3{\cdot}1416$, the ratio between the circumference of a circle and its diameter.

So that if the area of cross-section be denoted by A:

$$A = 3{\cdot}1416 \times (\text{radius})^2$$
$$= 3{\cdot}1416 \times \left(\frac{\text{diameter}}{2}\right)^2 = 3{\cdot}1416 \times \frac{d^2}{4} = {\cdot}7854\, d^2,$$

d being the diameter.

It is thus easy to calculate the areas of circular conductors if we know either the radius or diameter, and the sizes of wires could conveniently be given in terms of their areas. This is sometimes done; in other cases, the size is given in terms of the diameter, but more generally the size of a wire is given in terms of a *number*, known as the *gauge* of the wire. It is perhaps unfortunate and tends to ambiguity that several distinct wire gauges are in use—such as the B. W. G. (Birmingham Wire Gauge), the Board of Trade or S. W. G. (Standard Wire Gauge), commonly used in this country, and the B. & S. (Brown and Sharpe) or A. W. G. (American Wire Gauge), used in America—since the same number in the various gauges refer to wires of different sizes. For instance, it will be found by referring to tables of wire-gauges, that a wire 0·028 of an inch in diameter is 22 B. W. G., 22 S. W. G., and 21 B. & S., while a wire of gauge 10 is 0·134 of an inch, 0·128 of an inch, and 0·101898 of an inch according to the B. W. G., S. W. G. and B. & S. gauges respectively. It will also be noticed that as the number denoting the gauge increases, the size of the wire decreases.

There is still another method of designating the size of a wire which is of frequent use, and which might with advantage be used more universally, since the necessity of multiplying by the constant ·7854, as stated above, is to some extent removed in many problems in electrical engineering, when this method is adopted. In this system the number denotes the actual size of the wire, in terms of the *mil* standard, and this is an obvious advantage, since the introduction of π in the calculations is avoided. The *mil is a unit of length*, and is equal to one-thousandth of an inch (·001 inch). A circle, one

mil in diameter, has an area of one *circular mil*, which is ·7854 of a square mil. A circular mil is therefore the sectional area of a wire one mil in diameter.

Since
$$1 \text{ inch} = 1000 \text{ mils}$$
$$1 \text{ mil} = \cdot 001 \text{ inch}$$

a wire one mil in diameter has an area of cross-section of $\cdot 7854\,(\cdot 001)^2$ or ·0000007854 square inch, and an area of one circular mil,

$$\therefore \quad 1 \text{ circular mil} = \cdot 0000007854 \text{ square inch.}$$

The area of cross-section of any wire in circular mils is easily calculated since it is equal to the square of the diameter in mils, or

$$\text{area in circular mils} = d^2$$

d being the diameter in mils. Thus, a circle 2 mils in diameter has an area of 4 circular mils, and a conductor 3 mils in diameter has an area equivalent to the sum of the areas of nine wires, each one mil in diameter.

Then again Post Office authorities specify the size of iron wires for telegraphic purposes by the number of lbs. in weight per mile. An ordinary telegraph wire $\frac{1}{6}$th of an inch diameter weighs 386 lbs. per mile. It may also be mentioned that the British Post Office engineers specify the diameters of copper wires in mils.

Combining the expressions given in equations (6) and (7) we obtain the relationship between the resistance, dimensions, and material of a conductor:—i. e.

$$R = \frac{\sigma l}{A} = \frac{\sigma l}{\pi r^2} = \frac{4\,\sigma\,l}{\pi\,d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which σ = resistivity, l = length in centimetres, A = area of cross-section in square centimetres, r = the radius, and d = the diameter, both in centimetres. Equation (8) enables us to calculate very readily the resistance of a conductor when the resistivity and dimensions of a conductor are known, but it must be noted that the resistance calculated by means of this formula will be expressed in ohms or microhms according as σ is given in ohms or microhms.

Example I. A 20 B.W.G. wire is 0·035 of an inch in diameter; determine its area in circular mils.

Since 1 inch = 1000 mils

$\therefore$ 0·035 inch = 35 mils

and area in circular mils $= d^2$

$= (35 \times 35)$ circular mils

$= 1225$ circular mils.

Example II. The resistance of a mile of copper wire 92 mils in diameter is 6·2656 ω; determine the resistance of a mile of the same wire 42 mils in diameter.

Area of wire 92 mils diameter = 8464 circular mils.

Area of wire 42 mils diameter = 1764 circular mils.

And since resistance is *inversely* proportional to the area

circular mils circular mils

$$\therefore \quad 1764 \quad : \quad 8464 :: 6{\cdot}2656\,\omega : x \text{ ohms}$$

whence $$x = 30{\cdot}0635 \text{ ohms.}$$

Example III. Determine the resistance of a german-silver wire 4 metres long and 0.005 sq. cm. in sectional area, if σ for german-silver = 20·76 microhms.

Since $$R = \frac{\sigma \times l}{A}$$

$$\therefore \quad R = \frac{20{\cdot}76 \times 400}{0{\cdot}005} \text{ microhms}$$

and $$R = \frac{20{\cdot}76 \times 400}{0{\cdot}005 \times 10^6} \text{ ohms}$$
$$= 1{\cdot}66 \text{ ohms.}$$

§ **10. Conducting powers of materials. Conductance.** It will no doubt appear strange that whilst high conducting power is considered the essential property of a substance used as a medium for the conduction of electricity, a unit of the negative or converse property has been chosen for purposes of reference, instead of a unit denoting the facility with which a material permits the flow of electricity. We shall however presently point out that the obstruction of resistance offered by some materials is not only a convenience but an absolute necessity, and as the existing method has already taken deep root and is universally adopted, it would probably be a mistake to replace it by a more rational system dealing with the conducting property of a substance. As however the latter is more suitable for some problems frequently met with than the former, we shall now show how it may be usefully applied. *The property which a substance has for conducting electricity is termed conductance.* Conductance is the converse of resistance, and the conductivity of a material is the reciprocal of its resistivity; conductance is therefore measured in terms of a unit which is the reciprocal of the ohm, the unit of resistance. This unit is termed the *mho*, an inversion of the word ohm, that is *mho* is *ohm* written backwards, and was introduced by Lord Kelvin. Thus the mho is the conductance of a column of mercury 106·3 centimetres long, and one square millimetre in sectional area at 0° C., and it is evident that a conductor of R ohms resistance will have a conducting power of $\frac{1}{R}$ mhos. Conductance then is determined by dividing unity by the resistance in ohms. Consequently, as the resistance of a conductor increases conductance decreases, and conversely a substance of high conductance offers little resistance. As with resistance, conductance is a function of the dimensions, material and physical condition of the material, and varies according to the following laws.

(1) Conductance varies inversely as the length of the conductor, and if K denotes conductance, $K \propto \frac{1}{l}$.

(2) Conductance varies directly as the area of cross-section, that is $K \propto A$.

(3) Conductance varies as the specific conductance or conductivity of the material, and since conductivity is the reciprocal of resistivity, $K \propto \frac{1}{\sigma}$. Specific conductance is the conductance of a centimetre cube.

Example I. The resistance of a conductor 50 feet long is $5\,\omega$; determine the conductance of 40 feet of the same conductor.

$$\text{resistance of 40 feet} = \frac{4}{5} \text{ the resistance of 50 feet}$$

$$= \frac{4}{5} \times 5 \text{ ohms} = 4\,\omega$$

$$\therefore \quad \text{conductance} = \frac{1}{R} = \frac{1}{4} \text{ mho} = 0{\cdot}25 \text{ of a mho.}$$

Example II. The resistance of a certain conductor is $5\,\omega$. What will be the resistance of a conductor formed by taking two such conductors and placing them side by side, so that the new conductor will have twice the area of cross-section and the same length as the first? The solution of this problem is much simplified by dealing with the conductance instead of the resistance. In such problems the conductance increases with the number of wires placed side by side.

$$\text{The conductance of one wire} = \frac{1}{R} \text{ mho} = \frac{1}{5} \text{ mho}$$

$$\therefore \quad \text{the conductance of new conductor} = \frac{2}{5} \text{ mho}$$

$$\therefore \quad \text{resistance of new conductor} = \frac{5}{2} \text{ ohms} = 2{\cdot}5 \text{ ohms.}$$

In a later chapter we shall consider the various methods of connecting conductors in practice, and for the present shall only refer to some very simple cases. It will be obvious from what has already been said that the total resistance of a number of separate conductors connected in series will be equal to the *sum* of the separate resistances; if the same number of conductors are connected in parallel—i. e. connected so that they form one conductor by being placed side by side—the total conductance will be increased, and will be equal to the *sum* of the separate conductances.

Suppose there are n conductors the resistances of which are r_1, r_2 . . . r_n, then if connected in series the total resistance

$$R = r_1 + r_2 + r_3 + \ldots + r_n \quad \ldots \ldots \ldots \quad (9)$$

If connected in parallel the total conductance of the arrangement is

$$\text{conductance} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \ldots + \frac{1}{r} \quad \ldots \ldots \quad (10)$$

and since resistance is the reciprocal of conductance the joint resistance is

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots + \frac{1}{r_n}} \quad \ldots \ldots \quad (11)$$

When $n = 2$

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}$$

$$= \frac{r_1 r_2}{r_1 + r_2} \quad \ldots \ldots \ldots \ldots \quad (12)$$

Or, in words, the joint resistance of two wires connected in parallel is equal to *the product of the separate resistances divided by their sum.*

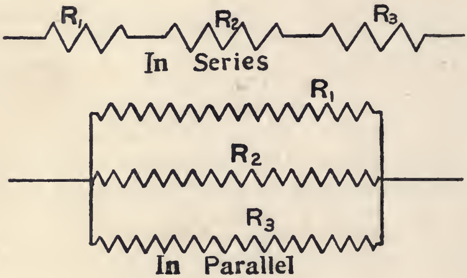


FIG. 2.

Example I. Three wires, the resistances of which are 40, 80, and 125 ohms respectively, are connected in parallel; what is the total conductance and joint resistance of the arrangement?

The separate conductances are $\frac{1}{40}$, $\frac{1}{80}$, and $\frac{1}{125}$ mho respectively,

and total conductance $= \frac{1}{40} + \frac{1}{80} + \frac{1}{125}$

$= \cdot 025 + \cdot 0125 + \cdot 008$

$= \cdot 0455$ mho

and joint resistance $R = \frac{1}{0 \cdot 0455} = 21 \cdot 978$ ohms.

Example II. Two conductors of 80 and 20 ohms resistance are connected in parallel, determine the joint resistance of the arrangement.

Since $R = \frac{r_1 \times r_2}{r_1 + r_2}$

$R = \frac{80 \times 20}{80 + 20} = \frac{1600}{100}$

$= 16$ ohms.

§ 11. Insulation. The unit of insulation Resistance. Whilst the resistance of a conductor should not be so great as to produce waste, or, in other words, whilst conductors should have so little

resistance that they form an easy path for the electricity passing, it is at the same time essential that means should exist for confining the electricity in a definite path, otherwise it will be found that it is the nature of electricity to take shorter paths so as to complete the circuit in a way not desired. As we have already mentioned Nature provides us with a plentiful supply of both conductors and insulators, giving us a large variety of materials which differ widely in their conducting powers, so that by making a judicious selection from these materials we are enabled to form *conductors*, permitting electricity to flow easily, and at the same time to enclose the conductors with *insulating materials* or *insulators*, giving us unlimited control over the current and the path, so as to prevent leakage, waste, and danger.

This point is very forcibly exemplified by the following abstract from the Presidential address of the late Sir W. Siemens, given before the British Association in 1882 :—'It is easy to confine the electric current within bounds, and to direct it through narrow channels of extraordinary length. The conducting wire of an Atlantic Cable is such a narrow channel; it consists of a copper wire, or strand of wires, 5 mm. in diameter, by nearly 5000 kilometres in length, confined electrically by a coating of guttapercha about 4 mm. in thickness. The electricity from a small galvanic battery passing into this channel prefers the long journey to America in the good conductor, and back through the earth, to the shorter journey across the 4 mm. in thickness of insulating material.'

As we have already said insulating materials have high resistances, and practically form *non-conductors* or *insulators*. The following is a list of the chief insulators: dry air, glass, porcelain, stoneware, slate, mica, ebonite, shellac, paraffin, resinous and bituminous compounds, oils, indiarubber, guttapercha, paper, silk, cotton, and wood. The specific resistances of these substances are so high, that we express insulation resistances in terms of a secondary unit, one million times the size of the ohm, which is termed a *megohm*. *One megohm equals* 1,000,000 *ohms*. The symbol for megohm is the capital Greek letter omega, Ω. Thus 100 Ω means 100 megohms.

Example. The insulation resistance of a thoroughly well insulated cable used in Electric Light circuits varies from 200,000,000 to 5,000,000,000 ohms per mile, that is from 200 to 5000 megohms per mile.

Insulation resistance is a function of the dimensions of the insulating covering of a cable, but with insulators the length of the insulating circuit is the *depth* or *thickness* of the covering, and the area of cross-section is proportional to the length of the covered cable. From this it is evident that if a mile of cable has an insulation

resistance of 1000 megohms, two miles will have 500 megohms, and five miles 200 megohms, and so on. Therefore to get the insulation resistance in megohms per mile—or the number of *megohm-miles*—the total insulation resistance of the cable is multiplied by the number of miles. The megohm-mile is a unit of resistance denoting the average insulation resistance of one mile of insulated cable. Insulation resistance varies considerably with atmospheric changes and changes in temperature, decreasing greatly with increase of temperature and addition of moisture.

Worked Examples. The determination of the resistance of a cable or conductor, or of the size of a conductor offering a certain resistance, is a problem frequently encountered by the engineer, and as it is so important a large number of typical examples will now be given to show how to use the data given or known. Let

R and R_1 be the respective resistances of two wires
l and l_1 ,, ,, lengths of these two wires
d and d_1 ,, ,, diameters ,, ,,
A and A_1 ,, ,, sectional areas ,, ,,
σ and σ_1 ,, ,, resistivities ,, ,,

then, assuming the temperature to remain constant, we have

$$\left.\begin{aligned} R &= \frac{\sigma \times l}{A} \qquad \text{I} \\ R_1 &= \frac{\sigma_1 \times l_1}{A} \qquad \text{II} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (8)$$

If the resistance of a conductor of given dimensions be known, we can very easily determine the resistance of another conductor of different dimensions, whether of the same material or not, provided the values of the specific resistances of the materials be known. To compare resistances we simply form a ratio between R and R_1 of (8), by dividing I by II, and we get

$$\frac{R}{R_1} = \frac{\sigma \times l}{A} \times \frac{A_1}{\sigma_1 \times l_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{A_1}{A} = \frac{\dfrac{\sigma}{\sigma_1} \times \dfrac{l}{l_1}}{\dfrac{A}{A_1}} \quad . \quad . \quad (13)$$

Expressing this in words, we get a useful formula for mentally comparing resistances, as follows :—

The ratio of the resistances

$$= \frac{\text{ratio of resistivities} \times \text{ratio of the lengths}}{\text{ratio of the sectional areas}}.$$

If the wires are made of the same material, then $\sigma = \sigma_1$ and (13) becomes

$$\frac{R}{R_1} = \frac{l\,A_1}{l_1\,A} \quad \ldots \ldots \ldots \quad (13\text{ a})$$

When the size of the wires is given in terms of the diameters d and d_1

$$\frac{R}{R_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{\cdot 7854\, d_1^2}{\cdot 7854\, d^2} = \frac{\sigma\, l\, d_1^2}{\sigma_1\, l_1\, d_2^2} \quad \ldots \quad (13\text{ b})$$

and
$$\frac{R}{R_1} = \frac{l\, d_1^2}{l_1\, d^2} \text{ if } \sigma = \sigma_1. \quad \ldots \quad (13\text{ c})$$

In some cases the cross-section of a wire has to be determined from the length and weight of the wire. Thus if W and W_1 be the weights of two wires whose lengths are l and l_1 and of specific gravities w and w_1, their sectional areas A and A_1 may be found as follows:—

Since w and w_1 are the specific gravities of the materials—i. e. the weights of unit volumes—it is evident that

$$W = \text{volume} \times w = l \times A \times w$$

and
$$\left.\begin{aligned} A \text{ or } \pi r^2 = \frac{W}{l \times w} \quad \therefore \quad r^2 &= \frac{W}{l \times \pi \times w} \\ \text{and} \qquad r &= \sqrt{\frac{W}{l \times \pi \times w}} \end{aligned}\right\} \quad \ldots \quad (14)$$

consequently
$$\frac{W}{W_1} = \frac{l \times A \times w}{l_1 \times A_1 \times w_1} = \frac{l \times r^2 \times w}{l_1 \times r_1^2 \times w_1}$$

and
$$\frac{A}{A_1} = \frac{l_1 \times w_1 \times W}{l \times w \times W_1} \quad \text{and} \quad \frac{r}{r_1} = \sqrt{\frac{W \times l_1 \times w_1}{W_1 \times l \times w}}.$$

If the wires are made of the same material, $w = w_1$

and
$$\frac{A}{A_1} = \frac{l_1}{l} \times \frac{W}{W_1}$$

and since
$$\frac{A}{A_1} = \frac{d^2}{d_1^2} \quad \therefore \quad \frac{d}{d_1} = \sqrt{\frac{l_1}{l} \times \frac{W}{W_1}}.$$

Again
$$R = \frac{\sigma \times l}{A} = \frac{\sigma \times l}{\pi r^2}$$

$$\therefore \quad R = \frac{\sigma \times l}{\pi \times (W/l \times \pi \times w)} \quad \text{from (14)}$$

$$= \frac{\sigma \times l^2 \times w}{W} \quad \ldots \quad (15)$$

To compare resistances when weights are given proceed as follows:—

$$\frac{R}{R_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{A_1}{A} = \frac{\sigma\, l}{\sigma_1\, l_1} \times \left(\frac{l \times w \times W_1}{l_1 \times w_1 \times W}\right) = \frac{\sigma}{\sigma_1} \times \frac{l^2}{l_1^2} \times \frac{w}{w_1} \times \frac{W_1}{W} \quad (15)$$

$$= \frac{l^2\, W_1}{l_1^2\, W} \text{ if the wires are made of the same material.}$$

It is not intended or recommended that all the above relationships should be committed to memory; they should, however, be studied

so that the fundamental formula, equation (8), which should be committed to memory, may be arranged to suit the problem under consideration. The following worked examples show how to do this.

Worked Examples. (1) A kilometre of No. 26 B.W.G. copper wire has a resistance of 105 ohms; what will be the resistance of a metre length of a cable made of ten of these wires placed side by side?

1 kilometre = 1000 metres

∴ 1 metre of No. 26 B.W.G. copper wire has 0·105 ohm resistance

ten of these wires placed side by side gives ten times the sectional area

∴ the resistance is $\frac{1}{10}$th of that of a single wire

∴ resistance of cable = $\frac{1}{10}$ of 0·105 = 0·0105 ohm.

(2) The resistance of a mile of copper wire 65 mils in diameter is 12·9 ohms. Determine the diameter of a wire of the same material, of which the resistance is 22·7 ohms per mile.

In the equation (13 b) $\frac{R}{R_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{d_1^2}{d^2}$

$R = 12{\cdot}9$ ohms; $R_1 = 22{\cdot}7$ ohms; $\sigma = \sigma_1$ and $l = l_1$

$$\therefore \quad d_1^2 = d^2 \times \frac{R}{R_1}$$

$$= (65)^2 \times \frac{12{\cdot}9}{22{\cdot}7} = (65)^2 \times 0{\cdot}5683$$

$$\therefore \quad d_1 = 65 \times \sqrt{0{\cdot}5683} = 65 \times 0{\cdot}754 = 49{\cdot}01$$

$$= 49 \text{ mils.}$$

(3) What are the relative resistances of two copper wires, one 0·134 inch in diameter and 670 feet long, and the other 0·067 inch in diameter and 134 feet long?

In $\frac{R}{R_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{d_1^2}{d^2} \qquad \sigma = \sigma_1$

$$\therefore \quad \frac{R}{R_1} = \frac{l}{l_1} \times \frac{d_1^2}{d^2} = \frac{670}{134} \times \frac{(0{\cdot}067)^2}{(0{\cdot}134)^2} = \frac{5 \times 1}{1 \times (2)^2}$$

$$= \frac{5}{4}$$

$$\therefore \quad R : R_1 = 1{\cdot}25 : 1.$$

(4) Determine the length of a copper wire 3·4 sq. mms. in area, which has the same resistance as an iron wire 100 yds. long and 6·04 sq. mms. in area. The conductivity of iron being taken as $\frac{1}{6}$th that of copper.

If l be the required length, then

$$l = l_1 \times \frac{\sigma_1}{\sigma} \times \frac{A}{A_1} \times \frac{R}{R_1} \text{ in which } R = R_1 \text{ and } \frac{\sigma_1}{\sigma} = 6$$

$$\therefore \quad l = 100 \times 6 \times \frac{3{\cdot}4}{6{\cdot}04} = 304{\cdot}635 \text{ yds.}$$

(5) It is found that 70 yds. of platinoid wire 25 mils in diameter has a resistance of $84\frac{1}{3}$ ohms at 0° C. Determine the specific resistance of platinoid in terms of (1) centimetre units, (2) 1 mil-foot units at 0° C.

Note. The specific resistance in centimetre units is the resistance of a centimetre cube, whilst the resistance of 1 foot of a wire 1 mil in diameter is termed the specific resistance in mil-foot units.

(1) Let σ be the specific resistance in centimetre units, then if l and A are in centimetres and square centimetres respectively

$$R = \frac{\sigma \times l}{A}$$

and
$$84\tfrac{1}{3} = \frac{\sigma \times 70 \times 12 \times 3 \times 2{\cdot}54}{{\cdot}7854 \times (0{\cdot}025 \times 2{\cdot}54)^2} = \frac{\sigma \times 70 \times 36}{{\cdot}7854 \times (0{\cdot}025)^2 \times 2{\cdot}54}$$

since 1 inch = 2·54 centimetres

$$\therefore \quad \sigma = \frac{84\tfrac{1}{3} \times (0{\cdot}025)^2 \times 2{\cdot}54 \times {\cdot}7854}{70 \times 36} = 0{\cdot}000041725 \text{ ohm}$$
$$= 41{\cdot}725 \text{ microhms.}$$

(2)
$$R = \frac{\sigma_1 \times l_1}{A_1}$$

where
σ_1 = specific resistance in mil-foot units
l_1 = length in feet
A_1 = area in circular mils = $(25)^2$ circular mils

$$\therefore \quad 84\tfrac{1}{3} = \frac{\sigma_1 \times 70 \times 3}{(25)^2}$$

and
$$\sigma_1 = \frac{84\tfrac{1}{3} \times 25 \times 25}{70 \times 3} = 251 \text{ ohms.}$$

(6) What is the resistance of 100 feet of wire weighing 4 grammes, if the resistance of another wire weighing 5 grammes and 75 feet long is 1·25 ohms, and its specific resistance $\frac{1}{10}$th that of the former? The specific gravity of the latter is 2·5 times that of the former.

Let R be the required resistance, A area of that wire, and A_1 the area of wire of known resistance, then

$$A = \frac{W}{l \times w} = \frac{4}{100 \times w}$$

and
$$A_1 = \frac{W_1}{l_1 \times w_1} = \frac{5}{75 \times 2{\cdot}5\, w}.$$

Substituting these and given values in the equation

$$\frac{R}{R_1} = \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{A_1}{A}; \text{ that is in } R = R_1 \times \frac{\sigma}{\sigma_1} \times \frac{l}{l_1} \times \frac{A_1}{A}$$

we get

$$R = 1{\cdot}25 \times \frac{1}{1/10} \times \frac{100}{75} \times \frac{5}{75 \times 2{\cdot}5\, w} \times \frac{100 \times w}{4} = \frac{100}{9}$$
$$= 11\tfrac{1}{9} \text{ ohms.}$$

(7) Given that the resistance of a copper wire of 98% conductivity 1 mil in diameter and 1 foot long has a resistance of 9·94 ohms, determine the percentage conductivity of a copper wire 1000 yds. long, 12 mils in diameter, which has a resistance of 215 ohms.

In this problem determine the resistance of the copper wire, assuming that it is made of the same material as that of the standard having a specific resistance of 9·94 ohms in mil-foot units, and compare with its actual resistance.

$$R = \frac{\sigma \times l}{A} = \frac{9{\cdot}94 \times 1000 \times 3}{(12)^2}$$
$$= \frac{2485}{12} = 207{\cdot}08\dot{3} \text{ ohms.}$$

$$\frac{12}{2485} \text{ mho} : \frac{1}{215} \text{ mho} :: 98 \text{ per cent.} : \text{percentage conductivity required}$$

$$\therefore \text{ percentage conductivity} = \frac{\text{conductivity of given wire} \times 98}{\text{conductivity of similar wire of standard}}$$

$$= \frac{\frac{1}{215}\text{ mho}}{\frac{12}{2485}\text{ mho}} \times 98$$

$$= \frac{2485 \times 98}{12 \times 215} = \frac{24353}{258}$$

$$= 94{\cdot}6 \text{ per cent.}$$

EXERCISES I B.

Electric Resistance.

(1) The conductance of a conductor is 2·5 mhos. What is its resistance?

(2) The resistance of a conductor is 25 ohms. What is the conductance of a conductor of the same material and cross-section, and half the length?

(3) The conductance of a wire 30 feet long is 4·5 mhos. What is the length of a wire of the same material and cross-section, the conductance of which is 3 mhos?

(4) The resistance of a wire 30 feet long is 4·5 ohms. What is the length of a wire of the same material and cross-section, the resistance of which is 3 ohms?

(5) A given wire has a resistance of 20 ohms. What would be the resistance of the wire doubled on itself (thus making a conductor of half the length and twice the area of cross-section)?

(6) Three lengths of wire are joined in series, i. e. 40 feet of wire having a resistance of 10 ohms per 100 feet, 25 feet having a resistance of 30 ohms per 100 feet, and 80 feet having a resistance of 25 ohms per 100 feet. Find the resistance of the arrangement.

(7) The resistance of 1 mile No. 10 B.W.G. copper wire being 3·128 ohms, calculate the united resistance of a quarter of a mile of nineteen No. 10 wires laid parallel, but not stranded. (C. and G.)

(8) Seventy-five inches of platinoid wire has a resistance of 5 ohms. How would you arrange 100 inches of this wire so that the resistance may be $1\frac{2}{3}$ ohms?

(9) Given that a column of mercury 106·3 cms. long and 1 square mm. in cross-section offers a resistance of 1 ohm. Determine the resistance of a column of mercury 280 cms. long and 0·8 square mm. in section.

(10) What is the resistance of a conductor 5 metres long and 1 millimeter in diameter, the specific resistance of the material being $1{\cdot}6 \times 10^{-6}$ ohms (1·6 microhms) per centimetre cube?

(11) Determine the resistance of a wire 1000 yards long and 125 mils in diameter if the mil-foot unit is 10·79 ohms.

(12) A piece of wire, 20 feet long and 10 mils in diameter, is found to have double the resistance of another wire of the same metal 15 feet long and of unknown diameter. What is the diameter of the second wire? (C. and G.)

(13) Two telegraph wires, each of the same length and of the same metal, have resistances of 2000 and 3000 ohms respectively. What are the relative diameters of the wires? (C. and G.)

(14) What are the relative resistances of two platinoid wires, one 25 mils in diameter and 500 feet long, and the other 24 mils in diameter and 480 feet long?

(15) What is the resistance of a copper conductor of specific resistance 1·598 microhms, $2\frac{1}{2}$ metres in length and 1 mm. in diameter?

(16) Determine the resistance of 125 yards of wire 75 mils in diameter, the resistance of 1 mile of that wire 24 mils in diameter being 0·01 ohm.

(17) A platinum wire of a certain length and cross-section has a resistance of 25 ohms. What would be the resistance of a platinum wire 25 times as long and $\frac{1}{5}$th the diameter of the first?

(18) The diameters of two copper wires are in the ratio 1 to 3. What will be the relative resistances of equal lengths? If two carbon rods are taken, one a metre long and a millimetre in diameter, and the other an inch long and a thousandth of an inch in diameter, what will be the relative resistances? 1 metre = 39·37 inches. (C. and G.)

(19) How long must a wire 0·045 inch in diameter be to have the same resistance as a wire of the same material 80 mils in diameter and 400 yards long?

(20) In a certain circuit it is necessary to replace a conductor 100 yards long and 0·48 inch in diameter with another conductor of the same metal 150 yards long without altering the resistance of the circuit. What size would you use?

(21) Determine the specific resistance of a wire 874 millimetres long and 0·5 millimetre radius, which has a resistance of 0·25 ohm.

(22) What is the resistance of a cube of silver whose side is 1 inch long, given that the specific resistance of silver is 1·504 microhms?

(23) What is the resistance of a conductor $2\frac{3}{4}$ yards long and 0·04 inch in diameter made of the material referred to in No. 21 above?

(24) What will be the diameter of a wire 250 feet long that has the same resistance as a conductor of the same material 25 mils in diameter and 360 feet long?

(25) You have three conductors, A, B, and C, of the same material given you, the radii of which are 10, 15, and 20 mm. respectively. What lengths of A and B would you take so that each would have the same resistance as 100 metres of C?

(26) A wire 250 yards long has $\frac{1}{5}$th of the resistance of a wire 600 feet long and 0·025 of an inch in diameter; determine the area of the first wire.

(27) What is the diameter of a wire 1 mile long which has a resistance of 20·25 ohms, if a mile of the same wire 45 mils in diameter has a resistance of 25 ohms?

(28) Platinum has six times the specific resistance of copper. What would be the relative diameters of two wires of these metals of equal lengths and equal resistances? (C. and G.)

(29) What must be the diameter of a copper wire which, taking equal lengths, gives the same resistance as an iron wire 6·5 millimeters in diameter, the specific resistance of copper and iron being 1·561 and 9·065 microhms respectively?

(30) A copper wire of known resistance is to be replaced by an iron wire of twice the diameter. What length of iron wire must be taken to have the same resistance as the copper?

(31) The resistance of 1 mile No. 10 B. W. G. pure copper wire is 3·128 ohms, and the diameter of this wire is 0·134 inch. Calculate the resistance of a quarter of a mile of german-silver wire 0·065 inch in diameter, having given that the specific resistance of german-silver is thirteen times that of copper. (C. and G.)

(32) How thick must an iron wire be so that half a mile of it shall offer the same resistance as a copper wire three-quarters of a mile long and 4·5 mils in diameter? The conductivity of iron is $\frac{1}{6}$th that of copper.

(33) Given that 1 mile of pure copper wire 0·134 inch in diameter has a resistance of 3·128 ohms. Calculate the resistance of 440 yards of a wire 0·067 inch in diameter, and having 95 per cent. conductivity. (C. and G.)

(34) The resistance of a sample of pure copper is 16·3 ohms, whilst the resistance of an exactly similar sample of commercial copper is 16·55 ohms. What is the percentage conductivity of commercial copper in terms of that of pure copper?

(35) Two exactly equal pieces of copper are drawn into wire, one into a wire 10 feet long, and the other into a wire 20 feet long. If the resistance of the shorter wire is 0·5 of an ohm, what is the resistance of the longer wire? (C. and G.)

(36) A wire 100 feet long has a resistance of 100 ohms. What would be its resistance if it were stretched to 105 feet? (C. and G.)

(37) A piece of copper wire 100 yards long weighs 1 pound; another piece of copper wire 500 yards long weighs $\frac{1}{4}$ pound. What are the relative resistances of the two wires? (S. and A.)

(38) Compare the resistances of two copper wires of the same weight, if the second one is ten times as long as the first.

(39) Two electromagnet bobbins are fully wound with the same weight of wire, the latter being 4 mils gauge in the one case and 2 mils gauge in the other; what will be the relative resistances of the two bobbins? (C. and G.)

(40) Two pieces of platinum wire are joined together so as to make a conductor 2·5 metres in length. The first part is 150 centimetres long and 0·12 millimetre in diameter, and the other part is 0·24 millimetre in diameter. The total resistance of the two wires is 14·08 ohms. Determine the specific resistance of platinum.

(41) Determine the specific resistance of copper, given that 1 foot of copper wire 1 mil in diameter is 10·233 ohms.

(42) Three telegraph wires A, B, and C, are successively looped together, and the resistance of each loop measured; the resistance of A and B is found to be 260 ohms, of A and C 280 ohms, and of B and C 300 ohms; determine the individual resistances of A, B, and C. (C. and G.)

(43) A platinum wire 1 metre long and weighing 1 gramme has a resistance of 1·96 ohms; what is the weight of a platinum wire 100 metres in length which has a resistance of 39·2 ohms?

(44) What will be the weight of a copper wire 100 yards long having a resistance of 1 ohm at 0° C.? Specific gravity = 8·9.

(45) Determine the specific resistance per centimetre cube of a wire 450 cms. in length which has a resistance of 1·25 ohms, if the specific gravity of the material is 8·9.

(46) Given that the specific resistance and density of aluminium are 2·665 microhms and 12·7 respectively, determine the resistance of an aluminium wire 2 metres long and weighing 1·5 grammes.

(47) Given that 1 foot of pure copper wire weighing 1 grain has a resistance of 0·2106 ohm, and the resistance of a conductor of commercial copper 5 feet long and weighing 5·75 grains is 0·9637 ohm. Compare the conducting power of commercial copper with that of pure copper.

(48) The resistance of a bar of iron 1 yard long, weighing 1 pound, is 0·00174 ohm. Calculate the resistance per mile of wires having the following weights:—400 pounds per mile; 200 pounds per mile; 150 pounds per mile. (C. and G.)

(49) What must be the relative diameters of wires of copper,

iron, and aluminium of the same weight so that their resistances may be equal?

(50) A kilogramme of copper of specific gravity 8·8 is to be drawn into a wire so as to offer a resistance of 100 ohms to a current of electricity. Determine its length and diameter given the specific resistance to be 1·621 microhms.

(51) The resistance of a certain wire l metres long and $(^1/_m)$th of a millimetre in diameter is R ohms. Determine the specific resistance of the material, and also the weight of the wire if its specific gravity is g.

(52) A wire changes its resistance from r ohms to R ohms by an increase of temperature of n degrees; construct a formula which will enable the change of resistance for any other change of temperature to be determined. (C. and G.)

(53) The resistance of a copper conductor is 25 ohms at 20° C. What is the resistance of this conductor at 15° C., given that the resistance of copper increases by 0·387 per cent. per degree Centigrade.

(54) Find the specific resistance of copper at 0° C., given that the resistance at 15° C. of a copper wire 1 metre long and 0·0376 cm. in diameter is 0·157 ohm and that its resistance increases by 0·387 per cent. per degree Centigrade.

(55) A coil of german-silver wire has a resistance of 100 ohms at 15° C. What is the range of temperature through which it may be used as a standard of resistance if the error must not exceed one-fifth of one per cent? Temperature coefficient $(\alpha) = 0{\cdot}00036$.

(56) At what temperature will a wire made of german-silver and 82·25 metres long and 0·25 sq. cm. in cross-section have a resistance of 1 ohm, if the specific resistance at 0° C. be 30 microhms and the temperature coefficient $(\alpha) = 0{\cdot}00036$?

(57) Having found by experiment that the resistance of a certain wire

at 0° C. is 1·6 ohms,
at 10° C. is 1·61 ohms,
at 15° C. is 1·615 ohms,

determine its resistance at 12° C.

(58) What is the resistance of a carbon filament 9 cms. long and 0·4 mm. diameter at 15° and 900° C.? Given that $\sigma = {\cdot}00409$ ohm, and $\alpha = 0{\cdot}0003$.

(59) The resistance of a copper cable is 1250 ohms at 10° C., what will be its resistance at 16° C. if the temperature coefficient $(\alpha) = 0{\cdot}00428$?

(60) The density of copper is 555 pounds per cubic foot, find the resistance of 5 pounds of copper wire of 20 S.W.G. (0·036 inch in diameter.)

(61) Find the lengths of manganin wire of the following sizes: 0·076, 0·16, 0·2006 millimetre respectively, which must be taken to form resistance coils of 1 ohm resistance, if the specific resistance of manganin is 43·75 microhms.

(62) What is the diameter of a wire 1000 feet long, the conductance of which is 0·0991 mhos, given that 1000 feet of the same wire 95 mils in diameter has a conductance of 0·87 mhos?

(63) Given that a single No. 16 wire has a resistance of 0·8 ohm per 100 yards, find the resistance of 75 yards of $\frac{7}{16}$ cable, if a stranded conductor has 3 per cent. more resistance than a solid conductor of the same cross-section.

(64) A certain coil is composed partly of copper wire of which $a = 0{\cdot}0039$, and partly of platinoid of which $a = 0{\cdot}0002$. The resistance of the complete coil at 0° C. is 1000 ohms, and at 20° C. it is 1010 ohms. Determine the resistances of the copper and platinoid wires separately at 0° C.

Section III. Electrical Pressure and Electromotive-force.

§ **12.** As we have already stated, all conductors have resistance and interpose obstruction to the passage of electricity; consequently, some force is required to overcome this resistance, and electrical pressure, upon which the transference of electricity depends, is much more important and difficult to understand than either current or resistance. We have already observed that electricity is a form of energy, and that bodies electrified are capable of doing work. In other words, electrification is accompanied by the production of a certain amount of energy, and any body brought into such a position or condition that in consequence it is capable of doing work, is said to possess potential energy. Electrically we say that a body electrified has a certain potential, which means that it possesses so much or so many units of potential energy. Now, potential is a measure or expression of the work which can be done, and work is done when electricity passes along a conductor, for then resistance (obstruction) is overcome, and it is found, and can be proved experimentally, that currents of electricity always result whenever we have two points in a conductor at different potentials, and means exist for the electricity to pass. An analogous phenomenon occurs when a body has two points at different temperatures; in which case

there is a transference of heat from the point of higher temperature to the point of lower temperature, and so long as this difference of temperature is maintained, there is a continuous flow or current of heat. Hence the following definition: *The difference of potential, or P.D., between two points, is that difference in electrical condition which tends to produce a transference of electricity from one point to the other.*

The letters P.D. (potential difference) are generally employed as an abbreviation of the term difference of potential. Now, if two bodies are electrified to different potentials, and they are connected by means of a wire, a rush of electricity takes place from the body of higher potential to the one of lower potential—that is, we have a discharge or momentary current, and, as the result, the two bodies are immediately reduced to the same electrical condition, and therefore to the same potential. To produce a continuous current it is evident that we must have something more than a P.D. We cannot at this point inquire into the process of transformation of any one form of energy into electrical energy, but must assume that every generator or source of electricity sets up something which maintains a P.D. This something is called 'electromotive-force,' and, as the name implies, it is a force in the sense that it sets up a pressure or stress which is a manifestation of electrical energy. The P.D. at the terminals of a dynamo or cell on open circuit is a measure of this stress.

In every circuit energy is expended, some usefully, in performing work, and some is dissipated by the resistance which has to be overcome. It is thus obvious that to continue this expenditure of energy, some external force (i. e. electromotive-force) is required to maintain a permanent P.D., and so keep up a continuous current of electricity, and any agency which enables this to be done is a source of electromotive-force. Hence the following definition:—*Electromotive-force is the originating cause or capability of maintaining a difference of potential, and is that which gives rise to an electric current in a circuit.*

The letters E.M.F. are usually employed as an abbreviation of the term electromotive-force. It is, however, important to remember that a source of E.M.F. does not create electricity, but is the active cause of electricity being in motion when a current of electricity traverses a circuit. It is thus clear that the relationship between E.M.F. and P.D. is the same as the relationship between force of gravity and water pressure or difference of water level. Both E.M.F. and P.D. are thus phases of energy, and it is because they are so intimately connected that so much confusion exists respecting them.

Throughout these pages the term electromotive-force (E.M.F.) will be used to denote the measure of the electrical pressure generated, and the term difference of potential (P.D.) to denote the measure of the electrical pressure existing between any two points in a circuit. The pressure utilized in a conductor, or external part of the circuit outside the generator, is the terminal P.D. of the generator, or, what is the same thing, the terminal P.D. is the fall of potential, or drop in volts, between the terminals. In hydraulic circuits, the analogous technical term, 'loss of head,' is used to denote the loss of water pressure overcoming the friction of the pipes.

Thus, if E be the E.M.F. of a generator, and e the terminal P.D., then

$$\mathrm{E} = e + e_1 \quad \ldots \ldots \ldots \quad (16)$$

e_1 being the electrical pressure used internally in the generator.

In the system of practical units, the unit of E.M.F. is that pressure which is required to send one ampere through a resistance of one ohm, and is called the 'volt.' This pressure is approximately the E.M.F. of a Daniell's cell. Unit work is done in a circuit when the coulomb or unit quantity of electricity traverses a circuit of unit resistance (1 ohm), or part of a circuit of unit resistance, and the volt is the measure of the capacity of a coulomb of electricity to do work. The terminal pressure or P.D. of a generator is usually termed voltage (number of volts), and the following statements will assist the student to distinguish between the different phases of electrical pressure. 'The term E.M.F. should be reserved for the energy supplied, while the destruction of energy can be denoted by voltage. The voltage can always be measured by a voltmeter; but E.M.F. cannot always be so determined. Voltage must be caused by E.M.F., but E.M.F. may exist without voltage. E.M.F. is generally local, while voltage is necessarily distributed. E.M.F. represents a condition of supply, which is generally fairly constant, however much power is supplied.' The E.M.F. of a circuit is independent of the resistance of the circuit, whilst, as we shall show later, the P.D. between two points in a circuit is proportional to the resistance between the two points.

§ 13. **The Board of Trade Definition of the Volt.** The legal definition of the Volt, the practical unit of electromotive-force as given by the Order in Council regarding units and standards of Electrical Measurements issued in 1894, is as follows:—

'The Volt, which has the value 10^8 in terms of the centimetre, the gramme, and the second of time, being the electrical pressure that if steadily applied to a conductor whose resistance is one ohm will

produce a current of 1 ampère, and which is represented by ·6974 $\left(\frac{1000}{1434}\right)$ of the electrical pressure at a temperature of 15° C., between the poles of the voltaic cell known as Clark's cell set up in accordance with the specification B, given below.'

Since the special form of voltaic cell called the Clark cell is the legal standard of electromotive-force, we give the specification

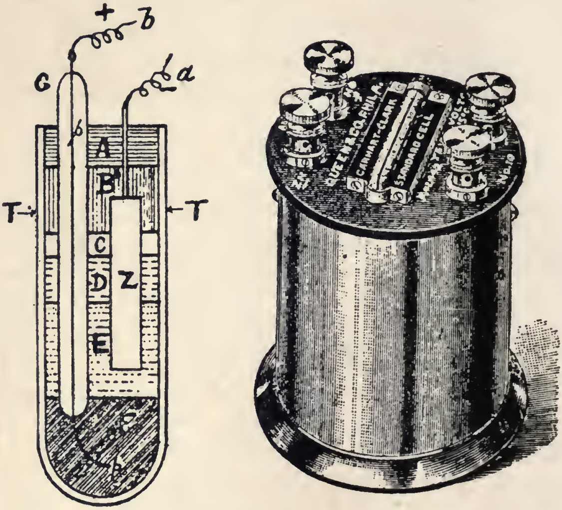


Fig. 3.

respecting the preparation of this cell in full, and diagrams of its construction in Fig. 3. It must also be noticed that the E.M.F. of this cell varies with the temperature, the percentage temperature coefficient being usually taken as 0·077 per degree Centigrade, and the E.M.F. at a temperature t° C. is given by the formula

$$E_t = 1{\cdot}434[1 - 0{\cdot}00077\,(t - 15)] \text{ volts.}$$

SPECIFICATION B.

On the Preparation of the Clark Cell.

Definition of the Cell.

The cell consists of zinc or an amalgam of zinc with mercury, and of mercury in a neutral saturated solution of zinc sulphate and mercurous sulphate in water, prepared with mercurous sulphate in excess.

Preparation of the Materials.

1. **The Mercury.**—To secure purity it should be first treated with acid in the usual manner, and subsequently distilled *in vacuo*.

2. **The Zinc.**—Take a portion of a rod of pure redistilled zinc, solder to one end a piece of copper wire, clean the whole with glass-paper or a steel burnisher, carefully removing any loose pieces of the zinc. Just before making-up the cell dip the zinc into dilute sulphuric acid, wash with distilled water, and dry with a clean cloth or filter paper.

3. **The Mercurous Sulphate.**—Take mercurous sulphate, purchased as pure, mix with it a small quantity of pure mercury, and wash the whole thoroughly with cold distilled water by agitation in a bottle; drain off the water, and repeat the process at least twice. After the last washing, drain off as much of the water as possible.

4. **The Zinc Sulphate Solution.**—Prepare a neutral saturated solution of pure ('pure recrystallized') zinc sulphate by mixing in a flask distilled water with nearly twice its weight of crystals of pure zinc sulphate, and adding zinc oxide in the proportion of about 2 per cent. by weight of the zinc sulphate crystals to neutralize any free acid. The crystals should be dissolved with the aid of gentle heat, but the temperature to which the solution is raised should not exceed 30° C. Mercurous sulphate treated as described in 3 should be added in the proportion of about 12 per cent. by weight of the zinc sulphate crystals to neutralize any free zinc oxide remaining, and the solution filtered, while still warm, into a stock bottle. Crystals should form as it cools.

5. **The Mercurous Sulphate and Zinc Sulphate Paste.**—Mix the washed mercurous sulphate with the zinc sulphate solution, adding sufficient crystals of zinc sulphate from the stock bottle to ensure saturation, and a small quantity of pure mercury. Shake these up well together to form a paste of the consistence of cream. Heat the paste, but not above a temperature of 30° C. Keep the paste for an hour at this temperature, agitating it from time to time, then allow it to cool; continue to shake it occasionally while it is cooling. Crystals of zinc sulphate should then be distinctly visible, and should be distributed throughout the mass; if this is not the case, add more crystals from the stock bottle, and repeat the whole process.

This method ensures the formation of a saturated solution of zinc and mercurous sulphates in water.

To set up the Cell.

The cell may conveniently be set up in a small test tube of about 2 centimetres diameter, and 4 or 5 centimetres deep. Place the mercury in the bottom of this tube, filling it to a depth of, say, ·5 centimetre. Cut a cork about ·5 centimetre thick to fit the tube; at one side of the cork bore a hole through which the zinc rod can pass tightly; at the other side bore another hole for the glass tube which covers the platinum wire; at the edge of the cork cut a nick through which the air can pass when the cork is pushed into the tube. Wash the cork thoroughly with warm water, and leave it to soak in water for some hours before use. Pass the zinc rod about 1 centimetre through the cork.

Contact is made with the mercury by means of a platinum wire about No. 22 gauge. This is protected from contact with the other materials of the cell by being sealed into a glass tube. The ends of the wire project from the ends of the tube; one end forms the terminal, the other end and a portion of the glass tube dip into the mercury.

Clean the glass tube and platinum wire carefully, then heat the exposed

end of the platinum red hot, and insert it in the mercury in the test tube, taking care that the whole of the exposed platinum is covered.

Shake up the paste and introduce it without contact with the upper part of the walls of the test tube, filling the tube above the mercury to a depth of rather more than 1 centimetre.

Then insert the cork and zinc rod, passing the glass tube through the hole prepared for it. Push the cork gently down until its lower surface is nearly in contact with the liquid. The air will thus be nearly all expelled, and the cell should be left in this condition for at least twenty-four hours before sealing, which should be done as follows :—

Melt some marine glue until it is fluid enough to pour by its own weight, and pour it into the test tube above the cork, using sufficient to cover completely the zinc and soldering. The glass tube containing the platinum wire should project some way above the top of the marine glue.

The cell may be sealed in a more permanent manner by coating the marine glue, when it is set, with a solution of sodium silicate, and leaving it to harden.

The cell thus set up may be mounted in any desirable manner. It is convenient to arrange the mounting so that the cell may be immersed in a water bath up to the level of, say, the upper surface of the cork. Its temperature can then be determined more accurately than is possible when the cell is in air.

In using the cell sudden variations of temperature should as far as possible be avoided.

The form of the vessel containing the cell may be varied. In the H form, the zinc is replaced by an amalgam of 10 parts by weight of zinc to 90 of mercury. The other materials should be prepared as already described. Contact is made with the amalgam in one leg of the cell, and with the mercury in the other, by means of platinum wires sealed through the glass.

Section IV. Ohm's Law.

§ 14. **Ohm's Law.** So far we have considered current, resistance, and E.M.F. individually, and without reference to their connexion with one another; but as these three fundamental factors are always present in an active circuit, we shall now endeavour to give an accurate idea respecting them in a collective sense. The relationship between them, and dependence upon each other, are conveniently expressed as a law, which, on account of its importance as the fundamental law of electrical measurements, may be considered in the strictest sense a law of nature. It was discovered and announced by Dr. G. S. Ohm, a German physicist, in 1827, in his *Mathematical Theory of the Galvanic Circuit*; and, in honour of the great service he thus rendered to the science of electricity, it is justly known as Ohm's Law.

From what has been said it is obvious that the strength of a current is proportional to the pressure acting, and inversely as the

resistance of the circuit. Ohm discovered that for the same conductors the ratio of the E.M.F. to the current, when steady, is constant, or

$$\frac{E}{C} = \frac{\text{cause}}{\text{effect}} = \text{a constant quantity}, \quad . \quad . \quad . \quad . \qquad (17)$$

where E = E.M.F. and C = current strength. This constant quantity was discovered by Ohm to be equal in all cases to the total resistance of the circuit; and subsequent experiments have so often verified this, that we say

$$\frac{E}{C} = R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

expresses in an algebraic or mathematical form the relationship known as Ohm's Law. It is clear that, given any two of the three quantities, we can, in a simple manner, determine the third, for by transposing one term we get $E = C \times R$, and by a further step we get $C = \frac{E}{R}$. This form is probably the most useful practically; but to those only slightly acquainted with algebraic statements, the following will, perhaps, be the most serviceable way of remembering Ohm's Law

$$\text{Amperage} = \frac{\text{voltage}}{\text{ohmage}},$$

where ohmage is used to denote the number of ohms resistance in the circuit. Expressed in words, Ohm's law is as follows:—*For a given conductor at a constant temperature the ratio between the electrical pressure and the resultant current, when steady, is a constant quantity; or the strength of the current, when steady, varies directly as the E.M.F. and inversely as the total resistance of the circuit.*

Collecting the various algebraic expressions for Ohm's Law, we have

$$C = \frac{E}{R}$$

or $$\text{current in amperes} = \frac{\text{electromotive force in volts}}{\text{resistance in ohms}}.$$

$$E = CR$$

or $$\text{E.M.F. in volts} = \text{current in amperes} \times \text{resistance in ohms}.$$

$$R = \frac{E}{C}$$

or $$\text{resistance in ohms} = \frac{\text{electromotive force in volts}}{\text{current in amperes}}.$$

The above relationships apply to any portion of a circuit as well as

to the complete circuit, and in a simple circuit composed of separate parts connected in series, the current strength is the same throughout, and from the equation $E = C \times R$ it follows that the difference of potential, P.D., between any two points, the resistance between which is r_1 ohms, is $e_1 = Cr_1$, and that the electromotive force of a circuit distributes itself proportionately to the resistance. The P.D., e_1, which is required to send the current C through the resistance r_1, is technically known as the 'drop' in volts in that part of the circuit.

If in any circuit there are several sources of electrical pressure, or several portions of the circuit differing in resistance, then E and R in the equation $C = \frac{E}{R}$ denote the sum of the pressures and resistances respectively, that is if e_1, e_2, e_3 = individual pressures, and r_1, r_2, r_3 = individual resistances: then

$$C = \frac{e_1 + e_2 + e_3 + \ldots + e_n}{r_1 + r_2 + r_3 + \ldots + r_n} = \frac{E}{R}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If the source consists of n cells, each having a pressure of e volts and r ohms resistance, then if the cells be connected in series

$$E = ne \text{ and } r_b \text{ (of battery)} = nr. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In addition to the resistance of the battery there may be the resistances of the leads, measuring instruments, lamps, &c. In the case of a simple circuit consisting of a battery of n cells, connecting wires and telegraph instrument, we have, for instance

$$C = \frac{E}{R} = \frac{n \cdot e}{r_b + r_w + r_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where r_b = total resistance of battery.

r_w = total resistance of leads or wires.

r_i = total resistance of instruments.

From equation (19) we obtain the relationship

$$E = C\{r_1 + r_2 + r_3 + \ldots + r_n\}$$
$$= Cr_1 + Cr_2 + Cr_3 + \ldots + Cr_n.$$

Now the products on the right-hand side of this equation are P.D.'s, or the pressures required to send the current through the separate portions of the circuit, and it is clear that the E.M.F. of the circuit is equal to the sum of the individual P.D.'s distributed throughout the circuit. In other words,

$$Cr_1 = e_1\,; \; Cr_2 = e_2\,; \; Cr_3 = e_3, \&c. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$C = \frac{E}{R} = \frac{e_1}{r_1} = \frac{e_2}{r_2} = \frac{e_3}{r_3} = \ldots = \frac{e_n}{r_n}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

These equations are algebraic statements of the important principles attending simple series circuits, i. e.

(1) The electromotive-force distributes itself uniformly along a conductor of uniform cross-section.

(2) The electromotive force distributes itself proportionally to the resistance.

Then again, since
$$\frac{E}{R} = \frac{e_1}{r_1}$$
$$\frac{e_1}{E} = \frac{r_1}{R}$$

or, in words, the P.D. between any two points in a circuit is the same proportion of the E.M.F. of the circuit that the resistance between the points is of the total resistance of the circuit. Again

$$\frac{e_1}{r_1} = \frac{e_2}{r_2}, \text{ \&c.}$$
$$\therefore \quad \frac{e_1}{e_2} = \frac{r_1}{r_2}.$$

Ohm's Law applies equally well to circuits in which all the electrical pressures do not act in the same direction; thus, in charging accumulators by means of a dynamo giving E volts a back pressure, e volts, is set up in the accumulators, and if we take the direction of the pressure of the dynamo as positive, then the direction of pressure set up in the cells will be negative, and the effective pressure sending the current through the circuit will be $(E - e)$ volts, and Ohm's Law in the algebraic form becomes

$$C = \frac{E - e}{R} \quad \quad (24)$$

Worked Examples. (1) A dynamo generating an electrical pressure of 120 volts sends a certain current through a circuit, the total resistance of which (including dynamo, lamps, &c.) is 10 ohms. What is the current?

Since $$C = \frac{E}{R}$$ E, R

$$\therefore \quad C = \frac{120 \text{ volts}}{10 \text{ ohms}} = 12 \text{ amperes.}$$

(2) If two cells of E.M.F.'s, 1·5 and 1·9 volts respectively, are connected so as to send a current through a resistance of 3·5 ohms, when the resistance of the cells are respectively 2 and 3 ohms, the resulting current is found as follows:

$$C = \frac{E}{R} = \frac{1\cdot5 + 1\cdot9}{2 + 3 + 3\cdot5}$$ E_1 E_2
$$= \frac{3\cdot4}{8\cdot5} = \cdot4 \text{ ampere.}$$

(3) Suppose that in the last example, the resulting current is too large, determine what additional resistance must be interposed in the circuit to reduce the current to 250 milliamperes.

Let r = the required resistance

then $\because$ 250 milliamperes = ·25 ampere

$$\cdot25 = \frac{3\cdot4}{8\cdot5 + r}$$

$$\therefore \quad 8{\cdot}5 + r = \frac{3{\cdot}4}{{\cdot}25} = 13{\cdot}6$$

and $$r = 13{\cdot}6 - 8{\cdot}5$$
$$= 5{\cdot}1 \text{ ohms.}$$

(4) Determine the terminal P.D., and the volts lost internally, in example (2) above.

If $$e = \text{terminal P.D.}$$
$$e_1 = \text{volts lost internally}$$

then $$\mathrm{E} = 3{\cdot}4 = e + e_1$$

but $$\frac{e}{\mathrm{E}} = \frac{r}{\mathrm{R}} (r = \text{external resistance})$$

$$\therefore \quad \frac{e}{3{\cdot}4} = \frac{3{\cdot}5}{8{\cdot}5} = \frac{7}{17}$$

and $$e = \frac{7}{17} \times 3{\cdot}4 = 1{\cdot}4 \text{ volts.}$$

Note. $$e = \mathrm{C} \times r = {\cdot}4 \times 3{\cdot}5 = 1{\cdot}4 \text{ volts}$$

$$\text{Lost volts} = e_1 = \mathrm{E} - e$$
$$= 3{\cdot}4 - 1{\cdot}4 = 2 \text{ volts}$$
$$= \text{pressure used internally.}$$

(5) A current of 25 amperes traverses a wire of which the resistance is 4 ohms per mile, determine the length of the conductor along which there is a drop of 2·5 volts.

Let $$l = \text{length of conductor in yards}$$

then $$\text{resistance of } l \text{ yards} = \frac{l}{1760} \times 4$$
$$\text{drop in volts} = e = \mathrm{C}r$$

$$\therefore \quad 2{\cdot}5 = 25 \times \frac{l}{1760} \times 4$$

$$1 = \frac{l}{44}$$

and $$l = 44 \text{ yards.}$$

(6) A dynamo, giving a constant voltage of 110 volts at its terminals, sends a current of 40 amperes when a certain external resistance is in circuit, and 25 amperes when a lamp is placed in series with the external resistance. What is the resistance of this lamp, and what is the drop in volts between its terminals?

Let r = the original resistance in the circuit when 40 amperes are passing, then

$$110 \text{ volts} = 40 \text{ amperes} \times r \text{ ohms}$$

$$\therefore \quad r = \frac{110}{40} = \frac{11}{4} \text{ ohms.}$$

If r_1 = resistance of the lamp added, then

$$110 \text{ volts} = 25 \text{ amperes} \times \left(\frac{11}{4} + r_1\right) \text{ ohms.}$$

and $$\frac{11}{4} + r_1 = \frac{110}{25} = 4{\cdot}4$$

$$\therefore \quad r_1 = 4{\cdot}4 - 2{\cdot}75 = 1{\cdot}65 \text{ ohms,}$$

and drop in volts

$$= e = \mathrm{C} \times r_1 = 25 \times 1{\cdot}65 = 41{\cdot}25 \text{ volts.}$$

(7) It is required to place a 100-volt lamp 100 feet from the street mains, the pressure of which is maintained at 110 volts. Determine the resistance of the connecting wire, given that the resistance of the lamp is 200 ohms.

Since the lamp requires 100 volts, the drop in volts for the connecting wire

$$= 110 - 100 = 10 \text{ volts};$$

the drop in volts for the connecting wire

$$= e = C \times r;$$

but the current required for the lamp

$$= \frac{100 \text{ volts}}{200 \text{ ohms}} = 0{\cdot}5 \text{ ampere}.$$

$$\therefore \quad 10 \text{ volts} = 0{\cdot}5 \text{ ampere} \times r \text{ ohms}$$

$$\therefore \quad r = \frac{10}{\cdot 5} = 20 \text{ ohms}.$$

It must be noted here that 20 ohms is the resistance of 200 feet of wire, i. e. 100 feet for lead, and 100 feet for return.

EXERCISES I c.

Ohm's Law.

(1) A battery of four Daniell's cells sends a current of 3 amperes through a certain circuit; if the E.M.F. of the battery is 4·4 volts, what is the total resistance of the circuit?

(2) What electrical pressure is required to send a current of 22 amperes through a resistance of 5 ohms?

(3) What current will a battery of five Grove's cells send through a resistance of 3·8 ohms? The E.M.F. of each cell is 1·9 volts.

(4) What currents will flow through a wire of 25 ohms resistance when the following potentials are maintained at its ends, (1) $+70$ and -30; (2) $+50$ and -50; (3) $+100$ and 0?

(5) A telegraph wire 50 miles long has a resistance of 12 ohms per mile, and an instrument of 400 ohms resistance is placed at its end; what current will ten Daniell's cells (of negligible resistance) send through the instrument? (C. and G.)

(6) A 10-cell Daniell's battery (E.M.F. 1 volt each), whose resistance is 10 ohms per cell, is used to work a telegraph line 8 miles long, the resistance of the line wire being 12 ohms per mile; determine the strength of the current in milliamperes, the resistance of the instrument being 200 ohms. (C. and G.)

(7) If, in the last example, it was required to reduce the current to 20 milliamperes, what additional resistance would you interpose in the circuit?

(8) The E.M.F. and internal resistance of a Daniell's, Grove's, and Bunsen's cell are respectively 1·1 volts, 0·5 ohm; 1·9 volts, 0·5 ohm; 1·7 volts, 1·5 ohms; they are connected in series. Determine the strength of the resulting current, if the external resistance of the circuit is 2 ohms.

(9) What would be the strength of the current if the Grove's cell were removed and replaced with its poles reversed?

(10) What would be the current in milliamperes sent by a 10-cell bichromate battery through a resistance of 100 ohms, each cell of the battery having a resistance of 5 ohms? E.M.F. of each cell 1·9 volts. (C. and G.)

(11) A telegraph line having a resistance of 100 ohms has a 10-cell battery connected to it, each cell having a resistance of 5 ohms; by how much would the current flowing be increased if the number of cells were doubled, the cells being joined up in series? (C. and G.)

(12) If $\frac{2}{3}$ of an ampere passes through a glow lamp which is placed between the mains of a 110-volt circuit, what is the resistance of the lamp when incandescent?

(13) Should the voltage of the above mains fall to 100 volts, and the resistance of the lamp be unchanged, what current will now pass through the lamp?

(14) A battery of six cells, each with an internal resistance of 0·4 ohm, sends a current of 2·85 amperes through an external resistance of 1·6 ohms. Determine the E.M.F. of each cell.

(15) By adding 2 ohms to the resistance of a certain circuit the current is diminished in the ratio of 5 to 4. What was the original resistance?

(16) By introducing an additional resistance of 28 ohms into a certain circuit the current falls from 2·1 amperes to 1·54 amperes, determine the E.M.F. acting in the circuit.

(17) If the increase of the resistance of a circuit by 10 ohms causes the strength of the current to decrease from 5 to 2, find the total resistance of the circuit after the change. (S. and A.)

(18) The terminals of a battery of five Grove's cells, the total E.M.F. of which is 9 volts, are connected by three wires, the resistance of each of which is 9 ohms. The current through each wire is $\frac{6}{7}$ of an ampere. What is the internal resistance of each cell? (S. and A.)

(19) Having given the length of a telegraph line, the resistance per mile of the wire, the resistance of the apparatus in circuit, and the number of milliamperes of current required to work the instruments, how would you calculate the number of Daniell cells required to work the line, allowing for a 50 per cent. loss of current through insulation leakage? The resistance of the battery cells may be assumed to be negligible. (C. and G.)

(20) A telegraph line, having (with the instruments in circuit) a total resistance of 2300 ohms, is required to be worked by Daniell cells, whose resistance per cell is 10 ohms; what number of cells would be required in order that the current transmitted through the line may be 25 milliamperes? Give the formula which will enable this to be worked out. (C. and G.)

(21) Twelve cells, 1·25 volts and 0·75 ohm each, are connected in series. What will be the current flowing through an external resistance of 16 ohms, if four of the cells are connected in opposition to the others?

(22) Twenty-five cells of internal resistance of 0·8 ohm, connected in series, send 0·4 ampere through a resistance of 50 ohms, how many of these cells also connected in series would produce the same current through each of two wires of 75 ohms connected in parallel?

(23) A battery of twelve equal cells in series screwed up in a box, being suspected of having some of the cells wrongly connected, is put into circuit with a galvanometer and two cells similar to the others. Currents in the ratio of 3 to 2 are obtained according as the introduced cells are arranged so as to work with or against the battery. What is the state of the battery? (S. and A.)

(24) How many volts are needed to send a current of 20 amperes through a conductor having 10 ohms resistance? What difference would it make in your answer if the circuit included a motor having a back electromotive-force of 450 volts? (C. and G.)

(25) A battery of four cells are connected in series and have an E.M.F. of 1·8 volts each. They are connected to two lead plates immersed in dilute sulphuric acid, and a current of 1·25 amperes traverses the circuit. If the total resistances of the circuit is 4 ohms, determine the back electromotive-force of the lead cell.

(26) A motor has a resistance of 2·5 ohms, and when an electrical pressure of 110 volts is maintained at its terminals, and the motor runs at such a speed that its back electromotive-force is 90 volts, determine the current flowing through the motor.

(27) A Daniell's cell, the internal resistance of which is 0·3 ohm, works through an external resistance of 1 ohm; what must be the resistance of another Daniell cell, so that when it is joined up in series with the first and working through the same external resistance the current shall be the same as before? (S. and A.)

(28) An electric motor is connected to a source of electricity giving a pressure of 20 volts, and 5 amperes traverse the motor when the armature is not allowed to rotate. When the motor is running at a certain speed the current passing is 2 amperes. What is the counter-E.M.F. set up in the motor?

(29) Take each quantity in the equation $e = CR$ in turn as a constant, and the others as co-ordinates x and y, and draw the resulting curves. What loci are obtained?

(30) Determine how the current varies with changes of external resistance with a battery of constant E.M.F. Show the result graphically.

(31) Determine how the voltage (P.D.) varies with the external resistance with a battery of constant E.M.F. Show the result graphically.

(32) A circuit consists of a battery of E.M.F. 5 volts and internal resistance 4 ohms, and three conductors of resistances 5, 10, and 12 ohms respectively connected in series. Determine graphically the current, terminal P.D., and fall in potential along each conductor.

Section V. Electrical Energy.

§ 15. Attention has already been drawn to the fact that electricity in motion is produced by the conversion of some one kind of energy into that of electrical energy, and the commercial applications of electricity consist in converting electrical energy into mechanical, thermal, or chemical energy. It is, therefore, of the utmost importance to be able to express in suitable units the work done electrically, and also to know the connexion between these units and the corresponding ones used to denote mechanical energy.

Now, whenever a current of electricity flows through a circuit, resistance is overcome and work is done, and in all cases energy is given up. We have already pointed out that energy is the capability of doing work, and that energy is measured by the number of units of work done. In mechanics, work is said to be done by a force overcoming a resistance through a distance, and it is well known that work is made up of two components: force and distance. Thus to raise 100 lb. of water vertically against the force of gravity work must be done, and it is evident that the work done is directly proportional to the force and also to the distance. Among engineers the unit of work is the foot-pound—that is, the work done by a force equivalent to the weight of 1 lb. acting through a distance of 1 ft. Work, then, is measured by the product of the force and the distance.

So far the time taken has not entered into the consideration of the work done. It is evident, however, that if it takes longer for one person to lift a certain weight through a certain distance than another person to do the same amount of work, then the power exerted in each case is not the same, although the amount of work done is the same. Work, then, is not the same as power. Power is the *rate* of working, being defined as the work done in unit time. The term 'activity,' suggested by Lord Kelvin, is now frequently used in the same sense, so that the activity in a circuit is the rate of doing work. The standard of power adopted by engineers is called the horse-power (H.P.), which is the name given to the rate of doing 33000 ft. lb. of

work per minute. The horse-power, then, of a machine is the number of foot-pounds of work done per minute divided by 33000, indicating the rate at which an engine or machine can do work. The relationship between work and power may be stated algebraically as follows:—

$$\text{work} = \text{power} \times \text{time}, \quad \text{and} \quad \text{power} = \text{work} \div \text{time}.$$

The principle of work is applicable to electrical energy, and to show the connexion between the above unit or rate of working and the unit of electrical power, we shall refer to the dynamo as a means of converting mechanical energy into electrical energy. And from what has been said previously it is known that any electrical effect must be attributed to the passage of electricity, and that an electric current only results whenever two points in a closed circuit are raised to, and maintained at, a difference of potential, so that when a certain amount of mechanical energy is expended in driving a dynamo and a current traverses the external circuit, it is evident that an electrical pressure has been set up; and, therefore, that current strength and electrical pressure (E.M.F.) are the components which make up the electrical energy given in exchange, as it were, by the dynamo for the mechanical energy supplied. At this point it may be mentioned in passing that a dynamo may be designed to give a small amperage at a high E.M.F., or a large amperage at a low E.M.F., and that the same mechanical power expended in driving a dynamo will produce 20 amperes at 50 volts, or 100 amperes at a pressure of 10 volts. And just as the product of the force and distance moved through per minute, or the number of foot-pounds per minute, measures the mechanical power expended, so the product of the number of volts and the number of amperes, or the number of volt-amperes, measures the electrical power generated, developed, or absorbed in any part of a circuit. Instead of the name 'volt-ampere,' originally used, the term 'watt' is now applied as the name of the unit of electrical power. That is

$$E\,(\text{volts}) \times C\,(\text{amperes}) = P\,(\text{watts}) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Again, the unit of electrical work is the joule; therefore, the watt may be defined as the joule per second. Now, the watt is one of the units most commonly met with in practice, and is usually defined as 'the power developed in a circuit when one ampere flows through it under a pressure of one volt.' And, although time is not mentioned in this definition, it is clear that this factor (time) is included, since an ampere is a definite quantity of electricity flowing per second. Therefore, to obtain the actual (electrical) work done in joules we have the relationship

$$J\,(\text{joules}) = E\,(\text{volts}) \times C\,(\text{amperes}) \times t\,(\text{seconds}) \quad . \quad . \quad (26)$$

If a dynamo were driven by a 1 H.P. engine, and no energy lost in heat, friction, leakage, &c., then we should receive back 1 H.P. of electrical energy, which, determined by taking the product of the volts and amperes given out by the machine, would be found to be 746 watts. Therefore

$$746 \text{ watts} = 1 \text{ H.P.} = 33000 \text{ ft. lb. per minute.}$$

Worked Examples. (1) A current of 15 amperes is sent through a circuit of 4 ohms resistance. What power in watts is absorbed?

Now, by Ohm's law $E = C \times R$

$$\therefore \quad \text{power } P = E \times C = (C \times R) \times C$$
$$= C^2 R$$
$$= (15^2 \times 4) = 900 \text{ watts.}$$

(2) Find the H.P. required to feed five hundred 110-volt lamps, the resistance of each being 220 ohms.

Power for each lamp $= E \times C$,

but $$C = \frac{E}{R} \text{ (by Ohm's Law)}$$

$$\therefore \quad \text{watts per lamp} = E \times \frac{E}{R} = \frac{E^2}{R}$$
$$= \frac{(110)^2}{220} = 55 \text{ watts,}$$

and $$\text{watts for 500 lamp} = 500 \times 55 = 27500$$

$$\therefore \quad \text{H.P. required} = \frac{27500}{746} = 37 \text{ (nearly).}$$

(3) If a 16 c.p. lamp requires 0·7 ampere at a pressure of 110 volts, how many watts does each candle-power require?

$$\text{Watts for each 16 c.p. lamp} = E \times C$$
$$= 110 \times 0{\cdot}7 = 77 \text{ watts}$$

$$\therefore \quad \text{watts per candle power} = \frac{77}{16} = 4{\cdot}8125 \text{ watts.}$$

(4) How many 16 c.p. lamps, taking 4 watts per candle, can be run off a dynamo giving 8 electric horse-power?

Since $$1 \text{ H.P.} = 746 \text{ watts}$$
$$8 \text{ H.P.} = 8 \times 746 = 5968 \text{ watts.}$$

And since 1 c.p. requires 4 watts

∴ each lamp of 16 c.p. requires 64 watts.

$$\therefore \quad \text{No. of 16 c.p. lamps} = \frac{5968}{64} = 93 \text{ lamps.}$$

(5) A dynamo is 100 yards from a house, the conductor has a resistance of 0·002 ohm per yard, and there are 150 30-watt 100-volt lamps to be fed. What E.M.F. must the dynamo give?

E.M.F. required = 100 volts for lamps + drop in volts along the mains.

$$\text{Now resistance of mains} = 0{\cdot}002 \times 200 \text{ (lead and return)}$$
$$= 0{\cdot}4 \text{ ohm.}$$

Note.—The conductor has to go *to* and *from* the house, and current required for each lamp is

$$C = \frac{P \text{ (watts)}}{E \text{ (volts)}}$$
$$= \frac{30}{100} = 0{\cdot}3 \text{ ampere.}$$

$\therefore$ current required for 150 lamps $= 0{\cdot}3 \times 150 = 45$ amperes.

And since the drop in volts along a conductor $= C \times R$,

the drop in volts through $0{\cdot}4\ \omega = 45 \times 0{\cdot}4 = 18$ volts.

$\therefore$ E.M.F. required $= 100 + 18 = 118$ volts.

§ 16. **The Kilowatt.** Another unit of power, adopted as a commercial unit, is the kilowatt, which is a secondary unit 1000 times the size of the watt, i. e.

$$1 \text{ kilowatt} = 1000 \text{ watts}.$$

And since the watt is $\frac{1}{746}$ of 1 H.P., it will be obvious that the value of the watt is extremely small, and, consequently, that large and somewhat inconvenient numbers have to be used to denote the power and size of large dynamos when given in watts. For instance, a dynamo supplying 500 amperes at a pressure of 120 volts gives out energy at the rate of (500×120) 60000 watts. Expressed in horse-power, this would be $\frac{60000}{746}$ or 80.4 H.P. And it is certainly more convenient to denote its power as 60 kilowatts, or its size as a 60-unit machine. A little consideration will show how the number of kilowatts can be easily turned into its equivalent number of horse-power. 1000 is slightly more than $1\frac{1}{3}$ times 746, therefore 1 kilowatt corresponds approximately to $1\frac{1}{3}$ H.P., from which we get the following rule:—

'The output of a dynamo in horse-power is obtained by adding one-third of the number of kilowatts to the number of kilowatts.'

Thus, in the above case, the output of the 60-unit machine

$$= 60 + \tfrac{1}{3} \text{ of } 60 = 60 + 20 = 80 \text{ H.P.}$$

§ 17. **The Connexion between Electricity, Heat, and Light.** Now that we are able to express in proper units the electrical energy equivalent to a certain amount of mechanical energy, we are in a position to consider the problem of converting electrical energy into heat and light. In a glow-lamp, for instance, a certain voltage and amperage are required to produce a certain amount of light—in other words, to keep a glow-lamp burning brightly, a certain number of watts of energy are consumed, and work is done at a *definite* rate. Now, the electrical power absorbed when electricity passes along a conductor is $C \times E$ watts, and if this energy is not transformed into mechanical or chemical energy, it is converted into thermal energy, either usefully for producing light, &c., or wastefully by being frittered away as heat. In either case the heat produced is equivalent to CE watts; but $E = C \times R$ by Ohm's Law, therefore the equivalent of the electrical energy as thermal energy is $(C \times R) \times C$, or C^2R watts. So that power may generally be defined as the rate of transformation

of one form of energy into another, and in an electrical circuit the heat produced per second is proportional to the number of watts consumed. And as a body becomes luminous and emits light when its temperature is very high, it follows that the light produced is proportional to the energy absorbed, so that glow-lamps giving the same light approximately absorb the same power. As is probably well known, the practical unit of light is the candle-power (c.p.), which is the amount of light produced by a standard candle weighing six to the pound, and burning 120 grains of spermaceti wax per hour.

Now, a well-constructed incandescent lamp absorbs from 3·5 to 4 watts per candle-power, from which it is evident that an ordinary 16 c.p. lamp of first-class make requires from 56 to 64 watts. It may be mentioned here, although we shall have to consider many interesting points respecting the heating of wires at the proper time, that it is usual and convenient to mark on the lamp the voltage required to send the necessary current through a lamp to make it burn brightly, and to designate the lamps 60, 100, 110, 200, or 220-volt lamps, as the case may be. If we know the voltage and amperage required by a certain lamp, the necessary horse-power to light a given number of lamps is found by dividing the product of the number of lamps, voltage, and amperage by 746, that is

$$\text{horse-power} = \frac{\text{N (No. of lamps)} \times \text{E (volts)} \times \text{C (amperes)}}{746}. \qquad (27)$$

§ 18. **Board of Trade Unit.** Since energy is stored-up work, it is measured in units of work, and as the joule is a small unit it is customary to express quantities of energy in *watt-hours.* When one watt is expended continuously for one hour a watt-hour of work is done.

$$1 \text{ watt-hour} = 3600 \text{ joules}.$$

Another unit—the legal quantity fixed by the Board of Trade for the purpose of public supply—i. e. the Board of Trade Unit, may now be considered. Technically it is called the *kilowatt-hour*, and as this name implies, it is the product of a rate of doing work into a period of time, and is the measure of an action lasting a certain time. Thus a consumer has received one B.O.T. unit, or one B.T.U., when the product of the electrical rate of working in watts and the number of hours during which the supply lasts equals 1000, and the number of B.T.U.'s is found by dividing the product of the volts, amperes, and hours by 1000. Thus ten 100-volt lamps taking one ampere each require a supply of one B.T.U. if placed in a circuit supplying energy at this rate for one hour. The following equivalents of the Board of Trade Unit are useful, and the student is advised to verify the same.

$$1 \text{ B.T.U.} = 1000 \text{ volt-ampere hours.}$$
$$= 3600000 \text{ joules.}$$
$$= 3600000 \text{ coulomb-volts.}$$
$$= 1{\cdot}34 \text{ H.P. working for one hour.}$$

The average charge by supply companies for electrical energy is 4*d.* per unit for lighting purposes, so that it is evident fifty glow-lamps taking 60 watts each would cost 1*s.* per hour; whilst if 3*d.* per unit be the charge for motive power, then the cost of an electrical horse-power for 1 hour is $2\frac{1}{4}d.$

Worked Examples. (1) What would it cost to light 250 220-volt 16 c.p. lamps for 100 hours, if the lamps take 60 watts each and the charge be 6*d.* per Board of Trade unit?

$$\text{Each lamp takes} = \frac{60 \text{ (watts)}}{220 \text{ (volts)}} = \frac{3}{11} \text{ ampere.}$$

And number of B.T.U.'s

$$= \frac{\text{N (No. of lamps)} \times \text{E (volts)} \times \text{C (amps.)} \times \text{T (hrs.)}}{1000}$$

$$= \frac{250 \times 220 \times \frac{3}{11} \times 100}{1000} = 1500.$$

$\therefore$ cost at 6*d.* per unit = 1500 × 6*d.* = £37 10*s.*

(2) A dynamo gives 40 amperes at 2000 volts. How long will it take to deliver one B.T.U.? What will be the income per hour if the charge be 6*d.* per unit?

The output of the machine $= \dfrac{40 \times 2000}{1000} = 80$ kilowatts = 107·2 H.P. and since there are 3600 seconds in 1 hour, the number of B.T.U.'s delivered per second $= \dfrac{80}{3600} = \dfrac{1}{45}$ B.T.U.

$\therefore$ 1 B.T.U. would be delivered in 45 seconds.

And since the output is 80 kilowatts

$\therefore$ in 1 hour 80 B.T.U.'s would be delivered,
and income per hour = 80 × 6*d.* = £2.

That is, 107·2 H.P. would be supplied for 1 hour for £2—that is at the rate of $4\frac{1}{2}d.$ per H.P. per hour.

(3) Determine whether it is more economical to use (*a*) 16-candle-power lamps taking 3 watts per candle and having a life of 600 hours, or (*b*) 16-candle-power lamps taking 3·5 watts per candle and having a life of 1000 hours. Given cost of electrical energy at 6*d.* per B.T.U., and cost of renewals at 1*s.* 3*d.* in each case.

The comparison may be made by assuming that the two kinds of lamps are used for 3000 hours, and the total cost of energy used and cost of renewals determined for each kind; but a better way is to determine the total cost per candle-hour for each lamp, as follows:—

(*a*) Total cost per candle-hour of lamps using 3 watts per candle.

Quantity of energy in B.T.U.'s per candle for 1 hour

$$= \frac{3}{1000} = 0{\cdot}003.$$

Cost of energy per candle-hour = 0·003 × 6 = 0·018 penny.

Cost for lamp renewals per candle-hour

$$= \frac{15}{16 \times 600} \text{ penny} = 0{\cdot}00156 \text{ penny.}$$

$\therefore$ Total cost per candle-hour = 0·018 + 0·00156 = 0·01956 penny.

(*b*) Total cost per candle-hour of lamps using 3·5 watts per candle.

Quantity of energy in B.T.U.'s per candle for 1 hour

$$= \frac{3{\cdot}5}{1000} = 0{\cdot}0035.$$

Cost of energy per candle-hour = 0·0035 × 6 = 0·021 penny.

Cost for lamp renewals per candle-hour

$$= \frac{15}{16 \times 1000} \text{ penny} = 0{\cdot}0009375 \text{ penny.}$$

$\therefore$ Total cost per candle-hour = 0·021 + 0·0009375 = 0·0219375 penny.

Therefore the lamp (*a*) is the more economical.

§ 19. Connexion between Electrical and Mechanical Units. Much confusion exists in the minds of many students respecting the various relationships existing between the foot-pound system of units and the centimetre-gramme-second (C.G.S.) system, and although the subject of units will receive full treatment in a later chapter it will be useful at this point to refer to some simple relationships connecting the magnitudes of force, work, and energy. For scientific purposes the C.G.S. system is to be preferred, since all magnitudes are referred to fundamental units based upon the centimetre as the unit of length, the gramme the unit of mass, and the second as the unit of time. The following definitions are important.

Unit force is that force which acting on unit mass for unit time will produce in it unit acceleration.

The C.G.S. unit of force is the *dyne*, and is that force which acting for 1 second on a mass of 1 gramme imparts to it a velocity of 1 centimetre per second.

The British unit of force is the *poundal*, and is that force which acting for 1 second on a mass of 1 pound imparts to it a velocity of 1 foot per second.

The weight of a body is that force with which the earth attracts the body, and if gravity be allowed to act on a mass during unit time the velocity acquired is g units. Therefore, if F denotes the weight of a body we have

$$F = m \, . \, g$$

where m is the mass of the body.

In the C.G.S. system 1 gramme-weight = g dynes.

In the British system 1 pound-weight = g poundals.

The force of gravity, however, varies in magnitude at different places, and consequently the value of g varies at different positions on the earth's surface. For the present we shall take g = 981 in the C.G.S. system, and g = 32·2 in the British system.

Unit work is done when unit force acts through unit distance.

The C.G.S. unit of work is the *erg*, and is the work done when the force of 1 dyne acts through a distance of 1 centimetre. The erg is a dyne-centimetre.

The British unit of work is the *foot-pound*, and is the work done when the force of the weight of 1 pound acts through a distance of 1 foot.

Since power is rate of working, the C.G.S. unit of power is the *erg-per-second* or *dyne-centimetre per second.*

The British unit of power is the *horse-power*, and is the rate of doing 550 foot-pounds of work per second, or 33000 foot-pounds of work per minute.

We may also mention here that

The joule = 10000000 or 10^7 ergs.
The watt = 10000000 or 10^7 ergs-per-second.
1 foot = 30·4797 centimetres.
1 pound = 453·59 grammes.

Worked Examples. (1) How many dynes are there in a poundal?

By definition

1 poundal gives a velocity of 1 ft. per sec. to a mass of 1 lb. in 1 second.

∴ 1 poundal gives a velocity of 30·4797 cms. per sec. to a mass of 453·59 grms.

∴ 1 poundal gives a velocity of (30·4797 × 453·59) cms. per sec. to a mass of 1 grm.

∴ 1 poundal gives a velocity of 13825·3 cms. per sec. to a mass of 1 grm.

But by definition

13825·3 dynes gives a velocity of 13825·3 cms. per sec. to a mass of 1 grm.

∴ 1 poundal = 13825·3 dynes.

(2) How many ergs are there in a foot-pound?

The weight of 1 lb. = the weight of 453·59 grammes
= g × 453·59 = 981 × 453·59 dynes
= 444971·79 dynes

∴ 1 foot-pound = 30·4797 × 444971·79 ergs.
= 13562606 ergs
= 1·356 × 10^7 ergs.

(3) How many joules are there in 1 foot-pound?

Since 1 foot-pound = 1·356 × 10^7 ergs
and 1 joule = 10^7 ergs
∴ 1 foot-pound = 1·356 joules
and 1 joule = $\frac{1}{1·356}$ foot-pound
= ·7372 foot-pound.

(4) How many foot-pounds are there in a kilogrammetre?

1 kilogrammetre = work done in moving 1000 grms. through 100 cms.
1 kilogrammetre = work done in moving 100000 grms. through 1 cm.
1 kilogrammetre = work done in overcoming 100000 × 981 dynes through 1 cm.
= 98100000 ergs,

but 1 foot-pound $= 1{\cdot}356 \times 10^7$ ergs.

$$\therefore \quad 1 \text{ kilogrammetre} = \frac{98100000}{1{\cdot}356 \times 10^7} \text{ foot-pounds}$$

$$= \frac{9{\cdot}81}{1{\cdot}356} = 7{\cdot}233 \text{ foot-pounds.}$$

(5) How many watts are there in 1 H.P.?

$$1 \text{ H.P.} = 550 \text{ foot-pounds per sec.}$$
$$= 550 \times 1{\cdot}356 \times 10^7 \text{ ergs per sec.}$$

but $$1 \text{ watt} = 10^7 \text{ ergs per sec.}$$

$$\therefore \quad 1 \text{ H.P.} = 550 \times 1{\cdot}356 \text{ watts}$$
$$= 746 \text{ watts (approx.).}$$

(6) A current of C amperes flows through a conductor of R ohms resistance when a pressure of E volts is maintained between the ends of the conductor; determine the number of foot-pounds of work done per minute.

$$\text{The power expended} = \text{C} \times \text{E watts}$$
$$= \frac{\text{CE}}{746} \text{ horse-power}$$

$$\text{and since} \quad 1 \text{ H. P.} = 33000 \text{ foot-pounds of work per minute}$$

$$\therefore \quad \text{the work done} = \frac{\text{CE}}{746} \times 33000 \text{ foot-pounds per minute}$$
$$= 44{\cdot}25 \text{ CE foot-pounds per minute}$$
$$= 44{\cdot}25 \text{ C}^2\text{R foot-pounds per minute}$$
$$= \frac{\text{CE}}{1{\cdot}356} \text{ or } \frac{\text{C}^2\text{R}}{1{\cdot}356} \text{ foot-pounds per second}$$

and $$1 \text{ watt} = \frac{1}{1{\cdot}356} \text{ foot-pounds per second}$$
$$= {\cdot}7372 \text{ foot-pounds per second.}$$

Electrical energy may be transformed into an equivalent amount of *thermal* energy, and in all cases some portion of the electrical energy of a circuit is frittered away as *heat*; for instance, wherever resistance is concentrated heat is manifested. The quantity of heat developed in a conductor by a current of electricity is a measure of the work done in that conductor, and it was found by Joule that the heat so produced is proportional to (1) the square of the current, (2) the resistance of the conductor, and (3) the time. We shall now consider the connexion between the thermal energy produced and the electrical energy producing it.

Def.—The unit of heat is that amount of heat which will raise 1 gramme of water through 1° C. of temperature, and is called the Calorie.

Joule also found that it requires 42000000 ergs of work to raise the temperature of 1 gramme of water one degree Centigrade; and this quantity, 42000000 ergs, which is equivalent to the calorie, that is the amount of mechanical work 1 unit of heat is capable of performing, is known as the mechanical equivalent of heat, and is often denoted by the letter J.

$$J = 42000000 \text{ ergs} = 1 \text{ calorie} = 4{\cdot}2 \text{ joules}$$
$$= 778 \text{ foot-pounds, when lbs. and degrees, F., are used.}$$

(7) How many foot-pounds are equivalent to 1 calorie?

Since $$1 \text{ watt} = \frac{1}{746}\text{H.P.} = \frac{550}{746}\text{ foot-pounds per sec.}$$
$$= 1 \text{ joule per sec.} = \frac{1}{4{\cdot}2}\text{ calorie per sec.}$$
$$\therefore \quad \frac{1}{4{\cdot}2}\text{ calorie} = \frac{550}{746}\text{ foot-pounds}$$
$$1 \text{ calorie} = 4{\cdot}2 \times \frac{550}{746} = 3{\cdot}09 \text{ foot-pounds}$$
$$= 3{\cdot}1 \text{ foot-pounds (approx.).}$$

(8) How many candles' illumination are developed per calorie in a 16 c.p. 100-volt lamp taking 0·6 ampere?

$$\text{Energy expended} = 100 \times 0{\cdot}6 = 60 \text{ watts}$$
$$= 60 \times 10^7 \text{ ergs per sec.}$$
$$= \frac{60 \times 10^7}{4{\cdot}2 \times 10^7} = \frac{10}{{\cdot}7}\text{ calories per sec.}$$
$$\therefore \quad \frac{10}{{\cdot}7}\text{ calories develop 16 candles' illumination.}$$
$$\therefore \quad 1 \text{ calorie develops } 16 \times \frac{{\cdot}7}{10}\text{ candles' illumination.}$$
$$\text{,,} \quad \text{,,} \quad 1{\cdot}12 \text{ candles' illumination.}$$

(9) A current of C ampères develops H units (calories) of heat in a conductor of R ohms resistance in t seconds. Determine the formula connecting H, C, R, and t.

Since J = amount of work in ergs performed by 1 calorie

$\therefore$ JH = amount of work in ergs performed by H calories,

and since C^2R = amount of power in watts expended per sec.

$\therefore$ C^2Rt = amount of work done in t seconds

$\therefore$ $JH = C^2Rt$.

But JH = number of C.G.S. (absolute) units of work.

$\therefore$ C^2Rt must be expressed in C.G.S. (absolute) units.

Now 1 absolute unit of current = 10 amperes (see Chapter XI),

and $$1 \text{ absolute unit of resistance} = \frac{1}{10^9}\text{ ohm}$$
$$1 \text{ absolute unit of E.M.F.} = \frac{1}{10^8}\text{ volt.}$$
$$\therefore \quad C \text{ amperes} = \frac{1}{10}C \text{ absolute units,}$$
and $$R \text{ ohms} = 10^9 R \text{ absolute units,}$$
and $$C^2Rt \text{ (practical units)} = \left(\frac{C}{10}\right) \times 10^9 R \times t \text{ absolute units.}$$
$$\therefore \quad JH = \frac{C^2}{100} \times 10^9 R \times t = C^2Rt \times 10^7$$
$$\therefore \quad H = \frac{C^2Rt \times 10^7}{J} = \frac{C^2Rt \times 10^7}{4{\cdot}2 \times 10^7}\text{ calories}$$
$$= \frac{C^2Rt}{4{\cdot}2} = C^2Rt \times 0{\cdot}24 \text{ calories}$$
$$= CEt \times 0{\cdot}24 \text{ calories,}$$

where H, C, R, E, and t are given in calories, amperes, ohms, volts, and seconds respectively.

(10) A current of 10 amperes flows through a resistance of 5 ohms for 6 seconds, and another current of 6 amperes through a resistance of 7 ohms; during what time must the latter current flow in order that the amount of heat generated in the two cases may be the same? (C. and G.)

Since $H = C^2Rt \times 0{\cdot}24$

$H_1 = 10^2 \times 5 \times 6 \times 0{\cdot}24$

$H_2 = 6^2 \times 7 \times t \times 0{\cdot}24$

where t = time required in seconds

but $H_1 = H_2$

$$\therefore \quad 6^2 \times 7 \times t \times 0{\cdot}24 = 10^2 \times 5 \times 6 \times 0{\cdot}24$$

$$36 \times 7 \times t = 500 \times 6$$

$$t = \frac{500 \times 6}{36 \times 7} = 11{\cdot}9 \text{ sec.}$$

(11) If one cell of an accumulator delivers 400 ampere-hours at an E.M.F. of 2 volts, calculate the energy in foot-pounds of this discharge, and the resulting temperature if all the heat generated be received by 10 kilogrammes of water at 31° C.

(*a*) Since the energy of discharge is measured by the work done, the energy = 44·25 CE foot-pounds per minute

and $\because$ 400 ampere hours = 400 amperes for 1 hour

$\therefore$ energy = 44·25 × 400 × 2 foot-pounds per minute
= 35400 foot-pounds per minute
= 35400 × 60 if continued for 1 hour
= 2124000 foot-pounds.

(*b*) Since $H = C \times E \times t \times 0{\cdot}24$ calories

heat generated = 400 × 2 × 60 × 60 × 0·24 = 691200 calories

and since 1 calorie will raise the temperature of 1 grm. of water 1° C.

$\therefore$ 691200 calories will raise the temperature of 10000 grms. 69·12° C.

$\therefore$ resulting temperature = 31° + 69° = 100° C.

EXERCISES I D.

Work and Power.

(1) A current of 15 amperes passes through a conductor for half an hour. If the P.D. at the ends of the conductor is 110 volts, determine the amount of work done in joules by the current.

(2) How many foot-pounds of work will be done in 20 minutes by a motor working at 4 H.P.? (C. and G.)

(3) How many joules of work will be done in 1 hour by a current of 15 amperes which traverses a circuit of 220 ohms resistance?

(4) At what rate is energy being supplied by a dynamo which supplies current for sixty 110-volt glow-lamps, connected in parallel, if each lamp takes 0·5 ampere?

(5) A current of 10 amperes flows through a copper wire of 100

ohms. At what rate is the energy being expended in the wire? (C. and G.)

(6) Compare the electrical power expended by a current, A, of 15 amperes under a pressure of 110 volts, and a current, B, of 25 amperes under a pressure of 100 volts.

(7) Compare the power expended in lighting (1) a 110-volt lamp of 220 ohms resistance, and (2) a 100-volt lamp taking $\frac{2}{3}$rds of an ampere.

(8) How many horse-power is required to drive a current of 100 amperes through a resistance of $7\frac{1}{2}$ ohms? (C. and G.)

(9) How many (electric) horse-power are needed to drive a dynamo supplying 120 16-candle-power lamps if the lamps take half an ampere each, and are supplied at 120 volts? (C. and G.)

(10) A current of 15 amperes is sent through a circuit of 220 ohms resistance. What is the horse-power expended?

(11) A 16-candle-power lamp requires 0·7 ampere under a pressure of 110 volts, how many watts does each candle-power require?

(12) How many 16-candle-power lamps, taking 4 watts per candle, can be run off a dynamo giving 7 electric horse-power? (C. and G.)

(13) If a lead has a resistance of 1 ohm and is traversed by a current of 10 amperes, what is the energy wasted in it in horse-power? (C. and G.)

(14) When a certain current passes through a certain conductor of 0·3 ohm resistance, the waste of power is 120 watts. Find the drop in volts along the conductor.

(15) It is found that the current through an incandescent lamp is 0·75 ampere, and that the lamp absorbs $\frac{1}{10}$ H.P. What is the E.M.F. required to operate the lamp? What will be the resistance of the lamp when working? (C. and G.)

(16) If the P.D. at the terminals of an arc lamp is 50 volts, and the power absorbed is 1 H.P., find the resistance of the lamp.

(17) A lamp of 400-candle-power requires $2\frac{1}{2}$ watts per candle-power. How many amperes will it take at 100 volts?

(18) How many 16-candle-power lamps each absorbing 3·5 watts per candle can be run from a 16-unit machine? If the lamps require 110 volts what current does the dynamo supply?

(19) If the output of a dynamo is 180 kilowatts at 1500 volts, what current would you expect to get from it, and what would be its E.H.P.?

(20) A 100-volt lamp taking $\frac{2}{3}$rds of an ampere is connected to the mains by leads of 0·75 ohm resistance. What proportion does the energy wasted in the leads bear to that utilized?

(21) The same current passes through two pieces of wire of the

same material, the diameter of the first being four times that of the second. Compare the number of watts absorbed in equal lengths of the two wires.

(22) A certain current traverses a given conductor. If the current be halved, how must the resistance be altered so that the same amount of energy may be absorbed?

(23) Two conductors of resistances r_1 and r_2 are connected between two terminals (1) in series, and (2) in parallel. Compare the energy absorbed in the two cases, (*a*) when the two terminals are maintained at a constant P.D., and (*b*) when they are supplied with constant current.

(24) How many poundals are there in 1000000 dynes?

(25) How many ergs are there in a foot-pound?

(26) How many foot-pounds are there in a joule?

(27) How many calories are there in a kilowatt-hour?

(28) Prove that 1·34 H.P.-hour = 1 B.T.U.

(29) Prove that 1 calorie = 3·968 watt-hours.

(30) Prove that 1 joule = 0·7372 foot-pounds.

(31) How many ergs correspond to 1000 watt-hours?

(32) A current of C amperes flows through a conductor of R ohms resistance when a pressure of E volts is maintained between the ends of the conductor, prove that

$$\text{the work done} = 44{\cdot}25\ CE \text{ or } 42{\cdot}25\ C^2R \text{ foot-pounds per minute}$$
$$= \frac{CE}{1{\cdot}356} \text{ or } \frac{C^2R}{1{\cdot}356} \text{ foot-pounds per second.}$$

(33) A current of C amperes develops H units of heat in a conductor of R ohms resistance in t seconds. Prove that

$$H = C^2Rt \times 0{\cdot}24 \text{ calories}$$
$$= CEt \times 0{\cdot}24 \text{ calories.}$$

(34) How many dyne-centimetres are equivalent to 1 calorie?

(35) How many candles' illumination are developed per calorie in a 16-candle-power 100-volt lamp taking 0·6 ampere?

(36) A group of twenty glow-lamps are connected in parallel and supplied with current at 100 volts; if the resistance of each is 150 ohms, determine (1) number of watts, and (2) number of foot-pounds of energy per minute, that are expended.

(37) In a given circuit supplied with current at 120 volts the expenditure of energy is 2400 foot-pounds per second; determine the current.

(38) An arc lamp receives 12 amperes at 50 volts and gives a mean candle-power of 600: how many candles per horse-power is this equivalent to?

(39) A single cell of a secondary battery delivers 400 ampere-

hours at an E.M.F. of 200 volts. Assuming 1 H.P. = 746 watts, calculate the energy, reckoned in foot-pounds, of the discharge. (C. and G.)

(40) A current of 10 amperes traverses a wire of 100 ohms, calculate the rate at which energy is expended in the wire in (1) foot-pounds per second, and (2) ergs per second.

(41) Prove that a Board of Trade unit is equivalent to 3600000 joules.

(42) At 6*d.* per Board of Trade unit, how much about does it cost for electric energy to keep ten lamps, each giving 15 candles, glowing for 5 hours? (C. and G.)

(43) If there were twenty-five arc lamps arranged in series on a 12-ampere circuit, and they each received 50 volts at their terminals, how many Board of Trade units would be used during a 5 hours' run of all the lamps, and what would be the cost at 4*d.* per unit?

(44) If a glow-lamp takes $3\frac{1}{2}$ watts per candle, and a gas burner gives 3 candles per cubic foot of gas burnt per hour, what must be the price of a Board of Trade unit so that the cost of electric lighting may be the same as that of gas lighting with gas at 2*s.* 9*d.* per 1000 cubic feet, the cost of lamps, fittings, wiring, &c., being excluded? (C. and G.)

(45) If in charging an accumulator cell the E.M.F. increases uniformly from 2 to 2·25 volts in 12 hours at 10 amperes, and discharges from 2·2 to 1·95 volts in 9 hours at 12 amperes. Determine the energy put in and given out both in watt-hours and foot-pounds.

CHAPTER II

ELECTRIC CIRCUITS AND ELECTRICAL EFFECTS

Section I.—Electric Circuits or Conductors in Practice.

§ **20. Electric Circuits.** We are now in a position to consider the various methods of arranging conductors in practice, and to consider how to calculate the resistances of the various combinations of conductors and the distribution of currents and potentials in such combinations. These calculations are important, and may be looked upon as the fundamental methods to be applied when dealing with the more difficult problems of distribution. As is well known, the complete path traversed by a current of electricity is termed an electric *circuit*, and it is obvious that a circuit must at least be composed of two parts: (1) the source or generator of electricity, and (2) the line or conductor connecting the terminals of the source. The former is known as the *internal circuit*, and the latter as the *external circuit.* In the external circuit, we usually have some receiving mechanism, transforming device, measuring instruments, &c.; and the general methods of arranging the separate portions (or elements, as they may be termed) composing a circuit are (*a*) the *series* arrangement, (*b*) the *parallel* arrangement, and (3) various combinations of the series and parallel arrangements.

'In series' is defined as that arrangement in which the positive terminal of an element is connected to the negative of the next, and so on, so that the whole thus connected forms a series of parts, through which the same current passes in succession, as shown in Fig. 4 *a*.

'In parallel' is defined as that arrangement in which all similar ends, terminals, or poles of the elements are connected to one conductor, and the other ends of the elements are connected to another conductor, so that the circuit is composed of several separate paths through which electricity may pass simultaneously, as shown in Fig. 4 *b*.

Fig. 4 illustrates these arrangements, and shows diagrammatically two circuits containing a resistance, lamps, and a measuring instrument. When the arrangement does not permit of electricity passing —as when the circuit is broken, for instance—the elements are said to be on *open circuit.*

§ 21. **Series Circuits.** Let r_1, r_2, r_3 ... r_n denote the several resistances of the individual portions or elements of a series circuit, and since the total resistances of such an arrangement must be the sum of the separate resistances

$$R = r_1 + r_2 + r_3 + \ldots + r_n$$

where R = total resistance. If there are n equal wires, or n parts having the same resistance connected in series, then $R = nr$. It is also obvious that the current will be the same at all points, and the

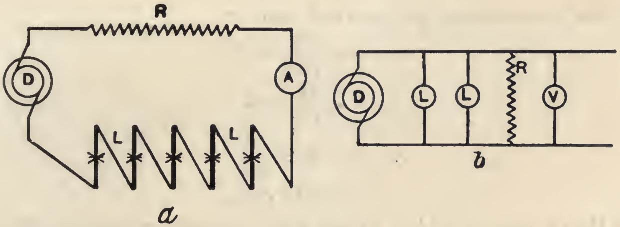


FIG. 4.

arrangement forms a constant-current circuit. Now, by Ohm's Law

$$C = \frac{E}{R} = \frac{E}{r_1 + r_2 + r_3 + \ldots + r_n}$$

$$\therefore \quad E = C(r_1 + r_2 + r_3 + \ldots + r_n) = Cr_1 + Cr_2 + Cr_3 + \ldots + Cr_n. \quad (28)$$

and the total E.M.F. of the circuit is equal to the sum of the individual P.D.'s required by the separate portions, since the product of the current and a resistance gives the electrical pressure required to send the current through the resistance. If we, therefore, denote the P.D.'s required by the separate portions by

$$e_1, e_2, e_3 \ldots e_n$$

then $\qquad Cr_1 = e_1, \; Cr_2 = e_2, \; Cr_3 = e_3,$

and so on, and

$$C = \frac{E}{R} = \frac{e_1}{r_1} = \frac{e_2}{r_2} = \frac{e_3}{r_3} = \frac{e_1 + e_2 + e_3 + \ldots + e_n}{r_1 + r_2 + r_3 + \ldots + r_n} \quad . \quad . \quad . \quad (29)$$

This is an algebraic statement of the fact that the electromotive-force distributes itself throughout a simple circuit uniformly, whilst in a series circuit the electromotive-force distributes itself propor-

tionately to the resistance. From the equation several useful relationships may readily be deduced. Thus, since

$$\frac{E}{R}=\frac{e_1}{r_1} \quad \therefore \quad \frac{e_1}{E}=\frac{r_1}{R} \quad \ldots \ldots \ldots \quad (30)$$

that is, the P.D. between any two points in a circuit is the same proportion of the E.M.F. of the circuit as the resistance between the points is of the total resistance of the circuit. Again

$$\frac{e_1}{r_1}=\frac{e_2}{r_2} \quad \therefore \quad \frac{e_1}{e_2}=\frac{r_1}{r_2} \quad \ldots \ldots \ldots \quad (31)$$

Worked Examples. (1) Fifty yards of wire, of which the resistance is 3 ohms per mile, is used to carry a current of 25 amperes, determine the drop in volts in the wire.

Drop in volts $= e = cr$

and $c = 25$ amperes

and since the resistance per mile is 3 ohms

$$r = \frac{50}{1760} \times 3$$

$$= \frac{15}{176} \text{ ohm}$$

$$\therefore \quad e = 25 \times \frac{15}{176}$$

$$= 2{\cdot}13 \text{ volts.}$$

(2) Half a mile of cable 1 square inch sectional area carries 250 amperes to and from a motor, if the P.D. at the terminals of the motor is 250 volts, determine the P.D. at the beginning of the circuit. A cubic inch of copper has 0·66 microhm resistance.

Let E = P.D. at the beginning of the circuit

$e = cr =$ drop in volts in the half-mile of cable

then $E = 250 + e.$

Now, $$r = \frac{\sigma l}{A} = \frac{0{\cdot}66}{10^6} \times \frac{880 \times 12 \times 3}{1}$$

$$= 0{\cdot}0209088 \text{ ohm}$$

and $$e = cr = 250 \times 0{\cdot}0209088$$

$$= 5{\cdot}2272 \text{ volts}$$

$$\therefore \quad E = 250 + 5{\cdot}2272$$

$$= 255{\cdot}2272 \text{ volts.}$$

(3) What percentage loss of potential is there when 50 amperes are sent through 100 yards of copper $\frac{1}{8}$ in. diameter, if the initial potential is 110 volts? A foot of copper wire 1 mil in diameter has 10 ohms resistance.

Since drop in volt or loss of potential $= e = C \times r$, and r, the resistance of 100 yards of $\frac{1}{8}$ in. copper

$$= 10 \text{ ohms} \times \frac{100 \times 3}{1} \times \frac{(\cdot001)^2}{(\cdot125)^2} = 0{\cdot}192 \text{ ohm}$$

$$\therefore \quad e = 50 \text{ (amperes)} \times 0{\cdot}192 \text{ (ohm)} = 9{\cdot}6 \text{ volts.}$$

And 110 volts : 100 volts :: 9·6 volts : p (percentage loss)

$$\therefore \quad p = \tfrac{96}{11} = 8\tfrac{8}{11} \text{ per cent.}$$

(4) Determine the diameter of copper bars employed to carry 4000 amperes

from a dynamo to a furnace, 10 yards apart, if the drop in volts in the bars is 0·1 volt.

Let r = resistance of the 20 yards of the bar (lead and return), then

$$0\cdot1 \text{ volt} = 4000 \times r$$

and $$r = \frac{0\cdot1}{4000} = 0\cdot000025 \text{ ohm}$$

$$\therefore \quad \text{resistance per inch of bar} = \frac{0\cdot000025}{20 \times 3 \times 12} = \frac{0\cdot0000025}{72} \text{ ohm.}$$

Now the resistance of a cubic inch of commercial copper is 0·00000066 ohm, and resistance varies inversely as sectional area

$$\therefore \quad \frac{0\cdot0000025}{72} \text{ ohm} : 0\cdot00000066 \text{ ohm} :: 1 \text{ sq. in.} : \text{A (area of bar)}$$

and $$\text{A} = \frac{0\cdot00000066 \times 72}{0\cdot0000025} \text{ sq. in.} = 19\cdot008 \text{ sq. in.}$$

But area of round bar $= \frac{\pi}{4} d^2$

$$\therefore \quad d = \sqrt{\frac{19\cdot008 \times 4}{\pi}} = 4\cdot918 \text{ in.}$$

(5) The P.D. between two points in a current is 100 volts, and a resistance coil is added to the circuit—but not between the two points—so as to lower the P.D. between the two points to 95 volts; given that the total resistance of the circuit at first is 57 ohms, and the total E.M.F. of the circuit 300 volts, determine the additional resistance required.

Let R = added resistance
r = resistance between the two points
C = current passing after the change

then $$\frac{100}{300} = \frac{r}{57} \text{ by equation (30)}$$

$$\therefore \quad r = \frac{57 \times 100}{300} = 19 \text{ ohms.}$$

Again $$300 = C \times (57 + \text{R})$$
and $$95 = C \times 19$$

$$\therefore \quad \frac{300}{95} = \frac{57 + \text{R}}{19} = 3 + \frac{\text{R}}{19}$$

$$\therefore \quad \frac{\text{R}}{19} = \frac{300}{95} - 3 = \frac{15}{95}$$

and $$\text{R} = \frac{15}{95} \times 19 = 3 \text{ ohms.}$$

It is obvious from this example that the P.D. between any two points in a circuit will be materially influenced by the addition or removal of resistances. We shall therefore give a general solution to this type of problem.

Let E = total E.M.F. of the circuit
R = total resistance of the circuit
e = original P.D. between any two points in the current of r ohms resistance
e_1 = the P.D. between the same two points when a resistance of r_1 ohms is added or removed from the circuit.

Now from equation (30)

$$\frac{e}{\text{E}} = \frac{r}{\text{R}} \, (\alpha) \text{ and } \frac{e_1}{\text{E}} = \frac{r}{\text{R}+r_1} \, (\beta)$$

and dividing α by β we get

$$\frac{e}{e_1} = \frac{R+r_1}{R} = 1 + \frac{r_1}{R}$$

$$\therefore \quad \frac{e}{e_1} - 1 = \frac{e - e_1}{e_1} = \frac{r_1}{R}$$

and $$r_1 = \frac{e - e_1}{e_1} \times R.$$

Note. $e-e_1$ is the change in the drop in volts between the two points.

(6) Determine the relationships existing between the E.M.F. of a battery, the P.D. at the battery terminals, the resistances of the different parts of the circuit and the current.

Let E = E.M.F. of the battery = the electrical pressure generated
e = the terminal P.D. of the battery = the electrical pressure spent externally
v = volts lost internally in the battery or source
r_b = internal resistance of the battery or source
r = external resistance
$R = r + r_b$ = total resistance
C = current traversing the circuit.

Then $$C = \frac{E}{R} = \frac{E}{r + r_b} = \frac{e}{r} = \frac{v}{r_b}$$

and $$E = C(r + r_b) = Cr + Cr_b$$

$$e = Cr; \quad v = Cr_b$$

$$\therefore \quad E = e + Cr_b = \frac{e}{r}(r + r_b)$$

and $$\frac{e}{E} = \frac{Cr}{CR} = \frac{r}{r + r_b}$$

$$\therefore \quad e = \frac{r}{r + r_b} \times E.$$

Also $$\frac{v}{E} = \frac{Cr_b}{CR} = \frac{r_b}{r + r_b}$$

$$\therefore \quad v = \frac{r_b}{r + r_b} \times E.$$

Again $$E = e + v = e + Cr^b$$

$$\therefore \quad e = E - v = E - Cr_b$$

and $$r_b = \frac{E - e}{C} = \frac{v}{C}.$$

(7) A conductor AC has a resistance of 10 ohms, and a point B is taken in it so that the resistance of BC is 4 ohms. This conductor is connected to a circuit so that the potential at A is 16 units, and that at C is 6 units. Determine the potential at the point B.

Let a = the potential at A
b = " " B
c = " " C
r_1 = the resistance of AB
r_2 = " " BC
R = " " AC $= r_1 + r_2$
C = the current flowing from A to C

then $$C = \frac{a - b}{r_1} = \frac{b - c}{r_2} = \frac{a - c}{R}$$

$$\therefore \quad (a-b)\,r_2 = (b-c)\,r_1$$

or
$$ar_2 - br_2 = br_1 - cr_1$$

and
$$b\,(r_1 + r_2) = cr_1 + ar_2$$

$$\therefore \quad b = \frac{cr_1 + ar_2}{r_1 + r_2} = \frac{cr_1 + ar_2}{R}.$$

Substituting the values given we have

$$b = \frac{6 \times 6 + 16 \times 4}{10} = 10 \text{ units.}$$

(8) The E.M.F. of a battery is 10 volts, and its terminal P.D. is e volts when a certain current traverses the external resistance of 12 ohms, determine (1) the value of e if the battery resistance is 8 ohms, and (2) the current which traverses the circuit when the external resistance is changed so that the terminal P.D. is 8 volts, assuming the battery resistance to remain constant.

(1) Since
$$\frac{e}{E} = \frac{r}{r + r_b}$$

$$\therefore \quad e = \frac{12}{12+8} \times 10 = 6 \text{ volts.}$$

(2) Since
$$c = \frac{v}{r_b} = \frac{E - e}{r^b},$$

$$c = \frac{10-8}{8} = 0{\cdot}25 \text{ ampere.}$$

(9) A certain battery sends 2 amperes through a circuit when the terminal P.D. is 12 volts, and 1 ampere when the resistance is changed so that the P.D. is 16 volts. Determine the internal resistance of the battery.

Since
$$E = e + v = e + cr_b$$

$$\therefore \quad \left.\begin{aligned} E &= 12 + 2\,r_b \\ E &= 16 + 1\,r_b \end{aligned}\right\}$$

and
$$12 + 2\,r_b = 16 + r_b$$

or
$$r = 4 \text{ ohms.}$$

EXERCISES II A.

Fall of Potential or Drop in Volts in Circuits.

(1) A battery of 10 ohms internal resistance and an E.M.F. of 12 volts sends a current through an external resistance of 20 ohms; what is the potential difference at the terminals of the battery?

(2) A battery of 5 ohms resistance has its poles connected by a resistance of 10 ohms. The P.D. between its poles is observed to be 15 volts; what is the E.M.F. of the battery? (C. and G.)

(3) A dynamo supplies a circuit with 75 amperes at a terminal voltage of 110 volts. If the internal resistance of the generator is $\frac{1}{15}$th of an ohm, determine the volts lost internally, and the E.M.F. generated.

(4) If a current of 20 amperes is flowing in a wire of which the resistance is 3 ohms per mile, state what will be the drop in volts for each 100 yards of this circuit. (C. and G.)

(5) A current is flowing along a copper wire, the current density being 1000 amperes per square inch. Find the fall of potential per yard, given that the resistance of 1 mile of copper wire 0·001 square inch section is 43 ohms.

(6) A certain cable was found to have a P.D. of 0·2 volt for every 100 metres of its length. The current traversing the cable was 25 amperes, determine the resistance of the cable per metre.

(7) A conductor ABC consists of two parts; the part AB is 1 metre long, 2 millimetres in diameter, and is made of copper of resistivity 1·6 microhms, whilst the other part BC is 1·25 metres long, 2·5 millimetres in diameter, and is made of iron of resistivity 9 microhms. If the potentials of A and C are 3 volts and zero respectively, find the potential of B, and the current flowing in the conductor.

(8) The terminals of a battery of twenty Daniell's cells connected in series are joined by a uniform wire 25 yards long, determine the position of a point in the wire which has the same potential as the eighth copper plate counted from the zinc end.

(9) A battery of internal resistance 5 ohms sends a current of 2 amperes when the external resistance is 3 ohms, determine the P.D. at the battery terminals, and also find the current which would flow if the external resistance is changed so that the terminal P.D. becomes 10 volts.

(10) The E.M.F. of a battery on open circuit is 12 volts, and the P.D. at the terminals is 10 volts when an external resistance is added and the circuit closed. Determine the internal resistance of the battery, and also the resistance of the wire if the current flowing is 5 amperes.

(11) A battery has a terminal P.D. of 14 volts when sending a current of 3 amperes, and 12 volts when sending a current of 3 amperes; determine the internal resistance of the battery and the E.M.F. of the battery.

(12) The E.M.F. of a battery is 18 volts, and its internal resistance 3 ohms. The difference of potential between its poles when they are connected by a wire A is 15 volts, and falls to 12 volts when A is replaced by another wire B. Calculate the resistance of A and B.

(13) A dynamo whose E.M.F. is 500 volts lights ten arc lamps connected in series, each of which has a resistance of 5 ohms. If the resistance of the leading wire is 0·75 ohm, and an ammeter in the circuit registers 9·6 amperes, determine the internal resistance of the generator, the terminal voltage of the dynamo, and the pressure required for each lamp.

(14) The E.M.F. of a battery is 10 volts and its internal resistance 4 ohms; the external circuit consists of three conductors connected in series having resistances of 3·5, 6, and 11·5 ohms respectively. Determine (1) the current; (2) the fall of potential along each wire; and (3) the P.D. at the battery terminals.

(15) A series dynamo having an internal resistance of 0·4 ohm maintains a terminal P.D. of 100 volts when supplying a certain current, if the volts lost internally is 17·5 per cent. of the E.M.F., determine the current and external resistance.

(16) A battery of 10 ohms resistance has its poles connected by a resistance of 5 ohms. The potential difference between the poles is observed to be 15 volts; what is the difference of potential lost in the battery?

(17) A battery giving an E.M.F. of 20 volts and an internal resistance of 15 ohms is sending a certain current through an external resistance of 5 ohms. Determine (1) the terminal P.D. of the battery; (2) the current traversing the circuit; and (3) the current which will traverse the circuit when the terminal P.D. is 8 volts.

(18) Determine the internal resistance of a battery which has a terminal P.D. of 10 volts when a current of 1·5 amperes traverses the circuit, and 8 volts when sending a current of $2\frac{1}{2}$ amperes.

(19) The E.M.F. of a battery is 15 volts; when the poles are joined by a wire the resulting current is 0·5 ampere, and the terminal P.D. is then 2·5 volts. Determine the internal and external resistances.

(20) A battery of ten cells connected in series, the E.M.F. of each being 1·8 volts, sends a current through a wire, and it is found that the P.D. between two points in the wire having a resistance of 12·5 ohms is 2·5 volts; determine the total resistance of the circuit.

(21) The internal resistance of a battery is five times the external resistance; determine the terminal P.D. if the E.M.F. is 15 volts.

(22) Determine the internal resistance of a battery, and also the external resistance, if, when 1·5 amperes is passing through the circuit, the terminal P.D. is 3 volts, and the E.M.F. is 15 volts.

(23) A dynamo giving a constant voltage of 110 volts at its terminals, sends a current of 40 amperes when a certain external resistance is in circuit, and 25 amperes when a lamp is placed in series with that external resistance. What is the resistance

of this lamp, and what is the drop in volts between its terminals?

(24) What must be the cross-section of a copper conductor 100 yards in length required to carry 15 amperes if the drop in volts is not to exceed 0·5 volt? The resistance of a cubic inch of copper may be taken as 0·66 microhm.

(25) What percentage loss of pressure occurs when 50 amperes are sent through 100 yards of copper wire $\frac{1}{8}$th of an inch in diameter, if this conductor forms part of a circuit the ends of which are maintained at P.D. of 110 volts? A foot of copper wire 1 mil diameter has 10 ohms resistance.

(26) AB is a conductor $1\frac{1}{2}$ miles long having a resistance of 40 ohms per mile. Two points C and D are taken in AB, such that C is $\frac{1}{3}$rd of a mile from A, and D is $\frac{1}{4}$th of a mile from B. If 110 volts are maintained between A and B, what is the percentage drop in volts between C and D?

(27) One part of a circuit ABC is shunted by a wire ADC, i. e. the two wires ABC and ADC are connected in parallel between the points A and C. The resistance of ABC is 5 ohms, and that of ADC 6 ohms. A short wire is connected to the point B, B being taken so that the resistance of AB is 2 ohms, find the position of the point D in ADC, such that on connecting the points B and D by the wire, no current passes along this wire.

(28) It is required to place a 100-volt lamp 100 yards from the mains, the pressure of which is 102 volts. Determine the cross-section of the connecting wires, given that the resistance of the lamp is $166\frac{2}{3}$ ohms, and that the resistance of a cubic inch of copper is 0·66 microhm.

(29) Six cells arranged in series, each having an internal resistance of 0·4 ohm, are connected by a wire of 1·6 ohms. If each cell has an E.M.F. of 1 volt, what is the potential difference between the positive pole of the battery and the point of junction of the third and fourth cells? (S. and A.)

(30) If the resistance of a cubic inch of copper is 0·66 microhm, and the diameter of each of a pair of wires be 0·064 inch, what length of circuit can be used so that the drop shall be 2 per cent. when the pressure between the wires at one end is 200 volts, and the current flowing through lamps connected between the wires at the other end is 2 amperes? (C. and G.)

(31) In the figure A is a dynamo supplying current to a circuit containing a lamp L and three resistances R_1, R_2, and R_3. The lamp requires 50 volts and has a hot resistance of 40 ohms. $R_1 = 5$ ohms, $R_2 = 10$ ohms, and $R_3 = 3$ ohms. Determine the current traversing

R_2, the drop in volts between a and b, and the terminal P.D. of the dynamo.

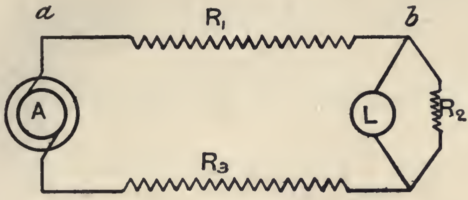


FIG. 5.

(32) A slide wire 105 centimetres in length and having a resistance of 1·2 ohms is connected in series with an adjustable resistance and a single accumulator cell. If the internal resistance of the latter be 0·1 ohm and its E.M.F. 2·1 volts, what value must the adjustable resistance have so that the drop along the slide wire may be exactly 0·01 volt per centimetre?

§ 22. **Parallel or Divided Circuits.** A very common method of forming circuits is to arrange two or more conductors so that their ends are practically connected immediately to the same two points, as at A and B in Fig. 6, in which three wires, having resistances of r_1, r_2, r_3 ohms respectively are connected to the two points A and B. In such cases it is evident that we have between the two points a *divided* circuit, and that the current will have the choice of three paths between the two points, with the result that the current will divide according to the power of the three branches to conduct electricity—i. e. according to the conductances of the wires. This arrangement is the same as that shown in Fig. 4 b, and is an example of a parallel circuit. Sometimes this arrangement is termed the *multiple-arc* arrangement.

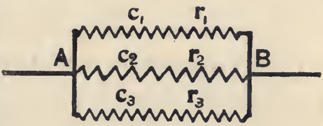


FIG. 6.

With the addition of separate paths forming a parallel or divided circuit, it is obvious that an increased area of conducting material is offered to the current—that is, the conductance between A and B increases with the number of branches, and the joint resistance of the wires diminishes. If we denote the separate conductances by k_1, k_2, and k_3, and the total conductance by K, then $K = k_1 + k_2 + k_3$; but conductance is the reciprocal of resistance.

$$\therefore \quad \frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

where R is the equivalent, combined or joint resistance of the wires between the points A and B.

$$\therefore \quad R = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3} \quad . \quad . \quad . \quad . \quad (32)$$

With only two branches, supposing the branch of resistance r_3 to be removed

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} \text{ and } R = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}} = \frac{r_1 r_2}{r_1 + r_2} \quad . \quad . \quad . \quad (33)$$

If $r_1 = r_2 = r_3$, then $R = \frac{r}{3}$, and generally for parallel circuits $R = \frac{r}{n}$, if r is the resistance of each branch, and n is the number of branches connected in parallel. To take a numerical example, let us find the joint resistance of three wires of 200, 500, and 1000 ohms resistance respectively, when connected in parallel (see Fig. 6). In this case

$$k_1 = \frac{1}{200}, \; k_2 = \frac{1}{500}, \text{ and } k_3 = \frac{1}{1000},$$

and $\frac{1}{R} = \frac{1}{200} + \frac{1}{500} + \frac{1}{1000} = 0{\cdot}005 + 0{\cdot}002 + 0{\cdot}001 = 0{\cdot}008$ mho.

$$\therefore \quad R = \frac{1}{0{\cdot}008} = 125 \text{ ohms.}$$

In this example it is obviously an advantage to express the separate conductances as decimals; in other cases it may be more convenient to make use of the expressions given in equations (32) and (33). A moment's consideration will suffice to show that when the branches are of unequal resistances, as in the numerical example above, the current divides through the branches unequally, and we shall now trace the relationship existing between the currents in the branches.

Let e denote the difference of potential between the points A and B (Fig. 6); then, if c_1, c_2, and c_3 denote the individual currents, we have, by Ohm's Law

√

$$e = c_1 r_1 = c_2 r_2 = c_3 r_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

√

$$\therefore \quad \frac{c_1}{c_2} = \frac{r_2}{r_1} \text{ and } \frac{c_2}{c_3} = \frac{r_3}{r_2}.$$

But (34) may be written

$$e = \frac{c_1}{\frac{1}{r_1}} = \frac{c_2}{\frac{1}{r_2}} = \frac{c_3}{\frac{1}{r_3}} \text{ or } \frac{c_1}{k_1} = \frac{c_2}{k_2} = \frac{c_3}{k_3} = e \quad . \quad . \quad . \quad (35)$$

and
$$c_1 : c_2 : c_3 :: k_1 : k_2 : k_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Expressed in words, we may say that the current divides itself among the branches of a parallel circuit in the ratio of the conductances, or inversely as the resistances of the branches. If the resistances are equal, the current divides equally. And in passing we may note that the general method of connecting incandescent lamps in practice is to place them in parallel across the mains, as shown in Fig. 4 *b*. If the lamps are the same candle-power and similar make, then

$$\mathrm{R} = \frac{r}{n} \quad \text{and} \quad c_1 = c_2 = c_3 = \&\text{c.} = \frac{\mathrm{C}}{n}$$

where n is the number of lamps, r the resistance of each, C the main current, and R the joint resistance of the lamps. Again, from (35)

$$c_1 = k_1 e\,; \; c_2 = k_2 e \quad \text{and} \quad c_3 = k_3 e$$

$$\therefore \quad c_1 + c_2 + c_3 = k_1 e + k_2 e + k_3 e$$

$$= (k_1 + k_2 + k_3)\, e$$

but $c_1 + c_2 + c_3 = \mathrm{C}$

and $k_1 + k_2 + k_3 = \mathrm{K}$ $\quad \therefore \quad \mathrm{C} = \mathrm{K}e.$

Again
$$\frac{c_1}{\mathrm{C}} = \frac{k_1 e}{\mathrm{K}e} = \frac{k_1}{\mathrm{K}}.$$

and
$$c_1 = \frac{k_1}{\mathrm{K}}\mathrm{C}$$

$c_1 = \frac{R}{r} C$

Similarly
$$\left.\begin{aligned} c_2 &= \frac{k_2}{\mathrm{K}}\mathrm{C} \\ c_3 &= \frac{k_3}{\mathrm{K}}\mathrm{C} \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (37)$$

Expressed in words, we may state that the current in any branch of a divided circuit is to the total current as the conductance of that branch is to the total conductance of the divided circuit.

In the case of a two-branch divided circuit, i. e. as when r_3 is removed, we have

$$\frac{c_1}{\mathrm{C}} = \frac{k_1}{\mathrm{K}} = \frac{k_1}{k_1 + k_2}$$

$$= \frac{\dfrac{1}{r_1}}{\dfrac{1}{r_1} + \dfrac{1}{r_2}} = \frac{r_2}{r_1 + r_2}$$

$$\therefore \quad \left.\begin{aligned} c_1 &= \frac{r_2}{r_1 + r_2}\mathrm{C} \\ c_2 &= \frac{r_1}{r_1 + r_2}\mathrm{C} \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (38)$$

Similarly

Divided circuits are very common in practice, and in many cases of electrical testing it is often convenient, if not necessary, to use

a conductor (in which there is no source of E.M.F.) connected in parallel with another conductor or instrument, so as to serve as a by-path, or *shunt* as it is called, to the latter conductor, as shown in Fig. 7, in which G is a delicate galvanometer, and S is the shunt.

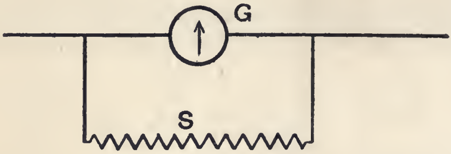


FIG. 7.

Since the arrangement is an example of divided circuit of two branches, the relationships already obtained hold, and if c_g is the current traversing the galvanometer of resistance, r_g, and c_s the current traversing the shunt of resistance, r_s, we have

$$\text{joint resistance} = \frac{r_s r_g}{r_s + r_g} \text{ from (33)}$$

$$\frac{c_g}{c_s} = \frac{r_s}{r_g} \text{ from (34)}$$

$$c_g = \frac{r_s}{r_s + r_g} C \quad \text{and} \quad c_s = \frac{r_g}{r_s + r_g} C \text{ from (38)}$$

$$\therefore \qquad C = \frac{r_s + r_g}{r_s} c_g$$

and the fraction, $\frac{r_s + r_g}{r_s}$, which is that quantity which must be multiplied by the galvanometer current, c_g, to give the total current in the main circuit, is termed the *multiplying power of the shunt*. Now from the relation, $c_g = \frac{r_s}{r_s + r_g} C$ it is evident that the current traversing the galvanometer may be made any desired fraction of the main current, or, in other words, may be varied within wide limits, by varying the resistance, r_s, of the shunt. Thus,

	if $r_s = r_g$	then $c_g = \frac{1}{2} C$
	,, $r_s = \frac{1}{2} r_g$	,, $c_g = \frac{1}{3} C$
	,, $r_s = \frac{1}{n} r_g$	,, $c_g = \frac{1}{n+1} C$
and	,, $r_s = \frac{1}{n-1} r_g$	,, $c_g = \frac{1}{n} C.$

To realize this property of a shunt in practice it is usual to employ shunt-boxes containing a number of resistance coils, arranged so that n may be 10 or some power of 10, and it is an easy matter to deter-

mine the value of r_s so that it may be related to r_g to give the desired result.

Thus when $n=10, \quad r_s=\frac{1}{10-1}r_g=\frac{1}{9}r_g$

$$n=100, \quad r_s=\frac{1}{99}r_g$$

$$n=1000, \quad r_s=\frac{1}{999}r_g$$

and so on.

Worked Examples. (1) Determine the total resistance of the circuit between AF (Fig. 8) formed of eight conductors arranged as shown.

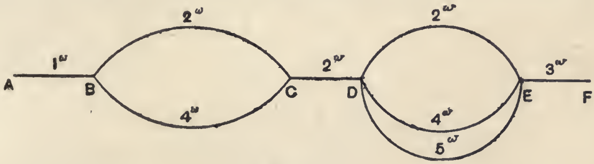


FIG. 8.

This is a complete arrangement of five parts connected in series, two parts of which, BC and DE, consist of conductors connected in parallel. Take the portion BC. The joint resistance of two conductors in parallel is

$$R=\frac{r_1 r_2}{r_1+r_2}$$

∴ joint resistance of the two conductors between the points B and C is

$$R_1=\frac{2\times 4}{2+4}=\frac{8}{6}=1\tfrac{1}{3} \text{ ohms.}$$

For the three conductors between the points D and E we have

$$\frac{1}{R_2}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$
$$=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{5}$$
$$=\tfrac{19}{20} \text{ mho}$$
$$\therefore \quad R_2=1\tfrac{1}{19} \text{ ohms.}$$

The total resistance between the points A and F is therefore

$$R=1+1\tfrac{1}{3}+2+1\tfrac{1}{19}+3 \text{ ohms}$$
$$=8\tfrac{22}{57} \text{ ohms.}$$

(2) Find the joint resistance of three 100-volt lamps giving respectively 8, 16, and 32 c.p. when placed between 100-volt mains, each lamp taking 3·75 watts per candle.

The lamps take respectively 30, 60, and 120 watts, and since watts absorbed $= EC = \frac{E^2}{R}$

$$\therefore \quad \frac{10000}{r_1}=30 \text{ and } r_1=\frac{1000}{3} \text{ ohms;}$$

similarly r_2 and r_3 for the 16 and 32 c.p. lamps are

$$r_2=\frac{1000}{6} \text{ and } r_3=\frac{1000}{12}$$

$$\therefore \quad \text{joint conductance } K = \frac{3}{1000} + \frac{6}{1000} + \frac{12}{1000}$$

and
$$\frac{1}{R} = \cdot 021 \text{ mho.}$$

$$\therefore \quad R = \frac{1}{\cdot 021} = 47\cdot 6 \text{ ohms.}$$

The same result may be obtained by considering the current passing. Thus the lamps take respectively 0·3, 0·6, and 1·2 amperes; therefore, total current used = 2·1 amperes. This current is due to a common P.D. of 100 volts overcoming the joint resistance of the lamps in parallel.

$$\therefore \quad R = \frac{e}{C} = \frac{100}{2\cdot 1} = 47\cdot 6 \text{ ohms.}$$

(3) The joint resistance of two wires connected in parallel is 4 ohms. If the resistance of one of them is 12 ohms, determine that of the other.

Since
$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\frac{1}{4} = \frac{1}{12} + \frac{1}{r_2}$$

$$\therefore \quad \frac{1}{4} - \frac{1}{12} = \frac{1}{6} = \frac{1}{r_2}$$

$$\therefore \quad r_2 = 6 \text{ ohms.}$$

(4) A wire of 10 ohms resistance is connected between mains carrying 2·8 amperes. What resistance must a wire have so that when connected in parallel with the wire of 10 ohms, 2 amperes will pass through the wire of 10 ohms?

The current obviously divides into two parts, 2 and 0·8 amperes respectively, and

$$\therefore \quad \frac{c_1}{c_2} = \frac{r_2}{r_1} \text{ by equation (34)}$$

$$\therefore \quad \frac{2}{0\cdot 8} = \frac{r_2}{10} \text{ and } r_2 = \frac{10 \times 2}{0\cdot 8} = 25 \text{ ohms.}$$

(5) A current of 1 ampere divides between three wires of 4, 8, and 10 ohms resistance respectively, connected in multiple arc. Determine the current in each branch.

The individual conductances are 0·25, 0·125, and 0·1 mho.

$\therefore$ total conductance = 0·25 + 0·125 + 0·1 = 0·475 mho.

$\therefore$ the currents in the separate branches are

$$c_1 = \frac{\cdot 25}{\cdot 475} \text{ of 1 ampere} = \frac{10}{19} \text{ ampere}$$

$$c_2 = \frac{\cdot 125}{\cdot 475} \text{ of 1 ampere} = \frac{5}{19} \text{ ampere}$$

$$c_3 = \frac{\cdot 1}{\cdot 475} \text{ of 1 ampere} = \frac{4}{19} \text{ ampere.}$$

If the current in the main current was C amperes, then c_1, c_2, and c_3 would be respectively $\frac{10}{19}C$, $\frac{5}{19}C$, and $\frac{4}{19}C$ amperes.

(6) An electromagnet which can safely take 5 amperes at an E.M.F. of 50 volts without heating injuriously, is connected across 110-volt mains; what resistance must a shunt connected to the terminals of the electromagnet

have, so that only 5 amperes traverse the coils of the electromagnet, if the connecting wires have 2 ohms resistance?

Let r_s = resistance of the shunt

and r = resistance of electromagnet coil

$$= \frac{50}{5} = 10 \text{ ohms}$$

$\therefore$ joint resistance of r and r_s

$$= \frac{10\,r_s}{10+r_s}$$

and the total resistance between the mains is

$$R = 2 + \frac{10\,r_s}{10+r_s}$$

but 50 volts = P.D. at the terminals of the electromagnet when 5 amperes traverse the electromagnet coils.

$$\therefore \quad \frac{50}{110} = \frac{\dfrac{10\,r_s}{10+r_s}}{2+\dfrac{10\,r_s}{10+r_s}}$$

$$= \frac{10\,r_s}{20+12\,r_s}$$

$$5\,(20+12\,r_s) = 11 \times 10\,r_s$$

and $5\,r_s = 10$

$\therefore \quad r_s = 2$ ohms.

(7) A galvanometer of 60 ohms resistance is shunted by a wire of unknown resistance so that the galvanometer is traversed by $\frac{1}{16}$ of the main current which is 0·8 ampere. Determine (1) the resistance of the shunt, and (2) the multiplying power of the shunt.

Since $c_g = \frac{1}{16}C$

$$r_s = \frac{1}{16-1}r_g$$

$$= \frac{1}{15} \times 60 = 4 \text{ ohms.}$$

And multiplying power of a shunt

$$= \frac{r_s+r_g}{r_s}$$

$$= \frac{4+60}{4} = 16.$$

(8) A galvanometer of 1980 ohms resistance is shunted by a wire of 20 ohms resistance, what amount of external resistance would have to be added in order that the insertion of the shunt may not change the total resistance of the circuit? Determine the currents traversing the galvanometer *without*, and *with*, this external resistance in the circuit, if the E.M.F. of the cell used is 1 volt, and the resistance of the cell and connecting wire is 0·2 ohm.

Joint resistance of galvanometer and shunt

$$= \frac{r_g \times r_s}{r_g+r_s}$$

therefore the resistance, r_g, between the terminals of the galvanometer (without the shunt) is reduced to $\frac{r_g\,r_s}{r_g+r_s}$ by inserting the shunt, therefore the

resistance of the circuit is diminished by $\left(r_g - \frac{r_g r_s}{r_g + r_s}\right)$ ohms

or $$\frac{r_g^2}{r_g + r_s} \text{ ohms,}$$

and this is the value of the external resistance which must be inserted so as to maintain the total resistance of the circuit. Substituting the values given, we have

$$\text{required external resistance} = \frac{(1980)^2}{1980+20}$$
$$= 1960{\cdot}2 \text{ ohms.}$$

Note. $1960{\cdot}2 = 1980 - \frac{1980 \times 20}{1980+20}$.

Currents traversing galvanometer

(*a*) without this external resistance in circuit

$$\text{Main current} = c_1 = \frac{1}{0{\cdot}2 + \frac{1980 \times 20}{1980+20}}$$
$$= \frac{1}{0{\cdot}2+19{\cdot}8} = \frac{1}{20} \text{ ampere}$$

and $$\text{galvanometer current } c_g' = \frac{20}{1980+20} \times c_1$$
$$= \frac{1}{100} c_1 = \frac{1}{100} \times \frac{1}{20}$$
$$= 0{\cdot}0005 \text{ ampere.}$$

(*b*) with external resistance in circuit

$$\text{Main current} = c_2 = \frac{1}{0{\cdot}2+1980}$$
$$= \frac{1}{1980{\cdot}2}$$

and $$\text{galvanometer current} = c_g'' = \frac{1}{100} c_2$$
$$= \frac{1}{100} \times \frac{1}{1980{\cdot}2}$$
$$= 0{\cdot}00005 \text{ ampere.}$$

EXERCISES II B.

Divided Circuits, and Currents in Divided Circuits.

(1) Determine the joint resistance of three wires of 250, 1000, and 500 ohms resistance respectively when connected in parallel.

(2) Determine the total resistance of the wires arranged as shown in Fig. 9.

(3) Determine the joint resistance of the seven wires arranged as shown in Fig. 10.

(4) A piece of wire is bent into the form of a square, and the ends are soldered together. If the resistance between the adjacent corners of the square be R, what is the resistance between opposite corners? (S. and A.)

(5) A wire, the total resistance of which is 4 ohms, is bent into the form of a square ABCD, the loose ends being soldered together. Find the resistance of the system, when a current enters at B and leaves at D. Will it be modified if the corners A and C be connected by another wire? (S. and A.)

(6) A length of uniform wire is bent into a circle of 1 foot radius, and two points A and B, a quarter of the circumference apart, are connected to a battery. Assuming that a foot of this wire has a resistance of 5 ohms, determine the joint resistance between A and B.

(7) The resistance of a copper wire 0·134 inch in diameter and 1760 yards long is 3·128 ohms. Calculate the resistance of 440 yards of a wire of the same metal 0·065 inch in diameter. What will be the joint resistance of these two wires if joined in parallel? (C. and G.)

(8) If 75 yards of $\frac{7}{16}$ cable are in parallel with 50 yards of $\frac{19}{20}$,

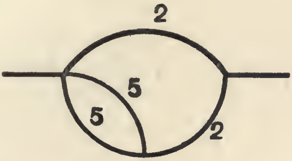


FIG. 9.

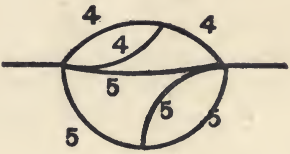


FIG. 10.

what is the joint resistance? A single No. 16 wire has a resistance of 0·8 ohm per 100 yards; a single No. 20 a resistance of 2·75 ohms per 100 yards; and a stranded conductor has 3 per cent. more resistance than a solid conductor of the same cross-section. (C. and G.)

(9) A cylindrical iron wire of 50 mils diameter is coated over uniformly with copper to a diameter of 75 mils. Calculate the resistance of 1000 feet of this compound wire, having given that 1 mil foot of copper has a resistance of 10·396 ohms, and that iron has a specific resistance seven times that of copper. (C. and G.)

(10) The joint resistance of two wires connected in parallel is $2\frac{2}{3}$ ohms. If the resistance of one of them is 8 ohms, determine that of the other.

(11) Two wires of 10 and 15 ohms respectively are connected in parallel between two points in a circuit. Determine the increase in the resistance of the circuit when the wire of 15 ohms resistance is removed.

(12) Six wires are joined in parallel between two points, three

have a resistance of 6 ohms each, the others having a resistance of 9 ohms each. Find the joint resistance of the arrangement.

(13) The resistance of two wires connected in series is $6\frac{1}{4}$ times the joint resistance of the same two wires connected in parallel. One of the wires has a resistance of 1 ohm, determine the resistance of the other.

(14) The resistance between two points AB of a conductor is 50 ohms, which is reduced to 20 ohms by joining a second wire to the points A and B; what is the resistance of the second wire?

(15) Two points in a circuit 25 yards apart have a wire of the same material, but $\frac{1}{10}$th its diameter and 5 yards long, connected to them so as to form a divided circuit; what proportion of the current passes in the two wires respectively?

(16) A wire of 10 ohms resistance is bent so as to form a square ABCD, the free ends being soldered together at A, and the two opposite corners A and C are connected to a battery. Compare the current that passes through ABC with that which passes through a piece of wire of the same kind that forms the diagonal of the square from A to C.

(17) Two wires of the same material whose lengths are l and l' and cross-sections s and s', respectively, are connected in parallel; compare the currents passing through the two wires, if the junctions are connected to a battery.

(18) Two wires are connected in parallel so as to form part of a circuit through which 15 amperes are passing. If 6 amperes pass through one branch, and the resistance of the other branch is 2 ohms, find the joint resistance of the arrangement.

(19) Four cells, each of E.M.F. 1·875 volts and internal resistance 0·5 ohm, are connected in series. Two wires ACB and ADB of 4 ohms and 12 ohms respectively are connected to the terminals A and B of the battery; determine the current strengths in the two wires.

(20) Two points A and B are connected by three wires APB, AQB, and ARB whose resistances are 1, 2, and 3 ohms respectively, and A is also connected to R, the middle point of ARB, by a wire ASR of 2 ohms resistance. How much of the total current flowing from A to B passes through each of the two branches between A and R? (London Univ.)

(21) The poles of a battery having an internal resistance of 1 ohm are connected by a wire of 9 ohms resistance. When an additional wire of 81 ohms is also connected between the two poles, prove that the current through the battery is increased in the ratio of 99 : 100, and that the current through the original wire is diminished in the ratio of 91 : 90.

(22) A current of 1 ampere divides between three wires of 2, 4, and 5 ohms resistance respectively, connected in parallel. Determine the current in each branch.

(23) A current of 28 amperes divides between three wires of 1, 2, and 4 ohms respectively, connected in parallel. Determine the current in each branch.

(24) Three wires of 6, 8, and 5 ohms respectively are connected in parallel. Give three numbers which will represent the strengths of the currents in the three wires, when a current enters at one junction. If 4 amperes pass along the wire of 6 ohms, determine the current in the main circuit.

(25) A wire of 25 ohms resistance is connected in parallel with another wire, and the two together carry 2·8 amperes. What must be the resistance of the second wire so that the wire of 25 ohms carries 0·8 of an ampere?

(26) An electric glow-lamp takes a current of ·75 ampere when working at 110 volts, calculate the hot resistance of 100 such lamps working in parallel. (C. and G.)

(27) A lamp circuit is supplied at a pressure of 100 volts. There are forty-two lamps all in parallel and the resistance of each lamp is (when hot) 150 ohms. How much current will the circuit take when all the lamps are on? (C. and G.)

(28) Two lamps of 100 and 150 ohms each, when running, are put in parallel with each other, and the pair are put in series with a resistance of 100 ohms.. What E.M.F. will be needed on the system if the current passing through the 100 ohm resistance is $1\frac{1}{4}$ amperes?

(29) What current will 15 cells, each having an E.M.F. of 1·07 volts and an internal resistance of 0·7 ohm, send through an electromagnet, having 28 ohms resistance, when the electromagnet is shunted with a wire having 7 ohms resistance? (C. and G.)

(30) A circuit is made up of (1) a battery with terminals A, B, its resistance being 3 ohms, and its E.M.F. 2·7 volts; (2) a wire BC, of resistance 1·5 ohms; (3) two wires in parallel circuit, CDF, CEF, with respective resistances 3 and 7 ohms; (4) a wire FA, of resistance 1·5 ohms. The middle point of the last wire is put to earth. Find the potential at the points A, B, C, F. (S. and A.)

(31) A wire having a resistance of 25 ohms per 100 feet is formed into a square, each side 30 feet long, and two adjacent corners are connected to the poles of a battery of ten Grove's cells (E.M.F. 1·9 volts and $r = 0{\cdot}3875$ ohm). Determine the current in the battery and in the two branches.

32) The joint resistance of two wires connected in parallel is

5 ohms, and the resistance of the longer wire is five times that of the shorter wire. Determine the resistances of the two wires.

(33) AB and CD in the figure are stout pieces of copper of negligible resistance. If $r_1 = 100$ ohms determine the value of r_2, so that the joint resistance of r_1 and r_2 is 50 ohms. Also determine value of r_3, so that joint resistance of r_1, r_2, and r_3 is 25 ohms. Similarly r_4, so that joint resistance of r_1, r_2, r_3, and r_4 is 12·5 ohms.

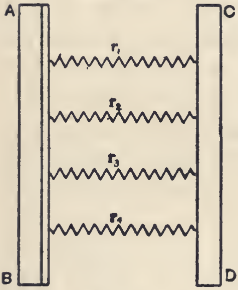


Fig. 11.

(34) It is required to use with a galvanometer of 5000 ohms resistance a shunt so that only 5 per cent. of the main current flows through the galvanometer. Determine the resistance of the shunt and the joint resistance of the galvanometer and shunt.

(35) The resistance of a galvanometer is 120 ohms. Find the resistance of a shunt so that $\frac{1}{5}$th of the whole current passes through the galvanometer.

(36) If $\frac{2}{5}$ths of the total current is to pass through a shunted galvanometer, what must be the resistance of a shunt compared with that of the galvanometer?

(37) In a certain circuit the galvanometer resistance is double that of the battery, and three times that of the shunt. Compare the battery currents and the galvanometer currents, (1) before, and (2) when using the shunt.

(38) A shunt of 10 ohms resistance is used with a galvanometer of 500 ohms, and a certain deflection is obtained with a battery of constant E.M.F. when the resistance of the rest of the circuit is 200 ohms. What additional resistance must be inserted to produce the same deflection when the shunt is removed?

(39) A galvanometer of 500 ohms resistance is shunted by a resistance of 100 ohms, what amount of external resistance would have to be added in order that the insertion of the shunt may not change the total resistance of the circuit? Obtain a formula which will enable the result to be determined. (C. and G.)

(40) Three lengths of cable of which the resistances are respectively 0·035, 0·025, and 0·013 ohm, are connected in parallel, and used to carry a current of 80 amperes. How much of this current flows through each of the three cables, and what is the potential difference between the two ends of the combination? (C. and G.)

(41) A glow-lamp, taking 0·5 ampere when supplied with 100 volts at its terminals, is connected with 100-volt constant pressure mains by means of two leads having together a resistance of $\frac{2}{3}$rds ohm.

What will be the current passing through this lamp when connected alone, and also when one, two, three, four, five, and six precisely similar glow-lamps respectively are turned on in parallel with the first, assuming that the resistance of the carbon filament is regarded as unchanged through the variation of the current? (C. and G.)

§ 23. **Insulation Resistance.** In practice insulators are employed to confine electricity along definite paths, and there are two ways in which insulating materials may be applied for this purpose. In the case of aerial lines, for instance, insulators are arranged as a series of supports attached to poles, and to these the conductors are bound. For this purpose the following dielectrics, porcelain, glass, india-rubber, &c., are moulded into appropriate forms with grooves, and the obstruction to the passage of leakage currents to earth (or the insulation resistance, as it is termed) is extremely high individually. The second method of insulating a conductor is to surround and envelop it with some high resistance dielectric from end to end, and is the method usually adopted in ordinary lighting circuits.

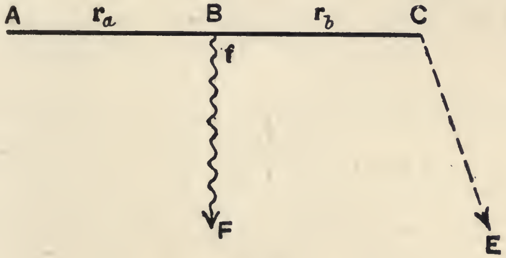


FIG. 12.

The insulation resistance in both cases is measured in megohms—one megohm being one million ohms—and a little consideration will show that all leakage currents are *in parallel* with one another, so that the determination of the insulation resistance of a circuit is an example of resistances connected in parallel. Speaking generally, the weak points of a circuit are formed by the connexions where joints are made, and switches, fuse-blocks, fittings are introduced, and it is at these points that leakage exists.

The laws of the divided circuit may conveniently be applied to determine the magnitude of insulation resistances, and the position of a fault, when one exists in a circuit, as shown below. Let ABC (Fig. 12) be a conductor insulated at A, B, and C, and if a = the insulation resistance of the portion AB, b = the insulation resistance of the portion BC, and x = the insulation resistance of the whole length AC, then since the insulation resistance of one section is in parallel

with that of a neighbouring section, the joint resistance of these sections gives the total insulation of the two sections, or

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} \text{ and } x = \frac{ab}{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

from which $$b = \frac{ax}{a-x}.$$

If the conductor resistance of the section $AB = r_a$, and that of $BC = r_b$, the total conductor resistance of $AC = r_x = r_a + r_b$, but if a partial fault occurs at B—i. e. the insulation resistance of the support at B is such that some current leaks to earth—of insulation resistance f, we have the following relationships.

(a) With the end C perfectly insulated the insulation resistance of the circuit is

$$r_1 = r_a + f \quad . \quad . \quad (\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

(b) With the end C earthed, the insulation resistance of the circuit is

$$r_2 = r_a + \frac{f \times r_b}{f + r_b} \quad . \quad . \quad (\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

since the earth branch BF, i.e. fault at B, is in parallel with the earth branch BC, but

$$r_x = r_a + r_b \quad \text{and} \quad f = r_1 - r_a \text{ [from } (\alpha)]$$

$$\therefore \quad r_b = r_x - r_a.$$

Substituting these values in (β) we have

$$r_2 = r_a + \frac{(r_1 - r_a)(r_x - r_a)}{(r_1 - r_a) + (r_x - r_a)}$$

$$= r_a + \frac{r_1 r_x + r_a^2 - r_a r_x - r_1 r_a}{r_1 + r_x - 2\, r_a}$$

and $$r_a^2 - 2\, r_a r_2 + r_x (r_2 - r_1) + r_1 r_2 = 0$$

from which $$r_a = r_2 \pm \sqrt{(r_2 - r_1)(r_2 - r_x)}.$$

Now from (β) r_2 is greater than r_a, therefore the negative root must be taken, and

$$r_a = r_2 - \sqrt{(r_2 - r_1)(r_2 - r_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

It is thus evident that if the actual conductor resistance, r_x, be known, then if a partial fault occurs at any point B, and tests be made with the end C perfectly insulated, and afterwards earthed, so as to obtain values for r_1 and r_2 as at (α) and (β), the conductor resistance r_a, of the portion AB, may be calculated, and the distance of the fault at B from A is given as a fraction of the length of the circuit AC; thus distance $AB = \frac{r_a}{r_x} \times$ length AC.

In the case of the insulating envelope of conductors the insulating material should possess the following properties in a high degree. (1) High value of resistivity to prevent leakage, since leakage means

waste of energy, possibility of electrolysis and danger from shock upon handling the cable. (2) Mechanical strength and be sufficiently tough to prevent abrasion. (3) Flexibility so as to bend without cracking. (4) Should resist the action of acids and gases, and be but little affected by increase of temperature; and (5) be homogeneous.

Although it is important that a cable should have a high insulation resistance, it is not an easy matter to calculate the insulation resistance of a cable, since the dielectric employed is usually of a composite nature, and the value of its resistivity difficult to evaluate. Then again, the variation of the resistivity of a dielectric with changes of temperature is much greater than with conductors, and it is important to remember that the resistivity of insulating materials diminishes at a rapid rate with an increase of temperature. If, however, the resistivity of the insulating envelope be known, and the geometrical relations of the conductor and the insulating covering be determined, an approximate value of the insulation resistance of a cable may be calculated. If D and d be the external and internal diameters of the insulating envelope, and σ the resistivity of the dielectric, it may be shown that for a length l

$$R_i = \frac{\sigma}{2{\cdot}728\, l} \log \frac{D}{d} = \frac{K}{l} \log \frac{D}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where R_i = insulation resistance, and K a constant depending upon the resistivity of the insulating material.

The insulation resistance of a cable obviously increases with an increase in the thickness of the insulating covering, and the Board of Trade rule requires that the insulation thickness shall not be less than $\frac{1}{10}$th of an inch for every 2000 volts. The Institution of Electrical Engineers' recommendations relating to insulation are important, and are as follows.

Insulated conductors may be broadly classed under two heads:—

A.—Those insulated with a material, as a dielectric, which is itself so impervious to moisture that it only needs further protection from mechanical injury or from vermin.

B.—Those insulated with a material, as a dielectric, which, in order to preserve its insulation qualities, must be kept perfectly dry, and therefore needs to be encased in a water-proof tube or envelope, generally of soft metal, such as lead, which is drawn closely over the dielectric.

When class A is used, the dielectric must be perfectly damp-proof, and not in any case less in thickness, measured radially, than 30 mils plus $\frac{1}{10}$th of the diameter of the conductor; it should not soften at a lower temperature than 170° Fahr. The minimum insulation of a test piece cut from it should be:—

1200 megohms per mile for conductors between 18 and 16 S.W.G.
800 megohms per mile for conductors between 16 and 14 S.W.G.
600 megohms per mile for conductors between 3/18 and 19/18 S.W.G.
400 megohms per mile for conductors between 7/14 and 39/16 S.W.G.
300 megohms per mile for conductors between 19/12 and 91/11 S.W.G.
the test being made at 60° Fahr., after one minute's electrification, and after the test piece has been immersed in water for 24 hours.

When class B is used, the same conditions as to minimum thickness and softening temperature of the dielectric should be enforced, as in class A; its covering should be such that a test piece cut from the conductor and immersed in water will not break down when an alternating pressure of 2500 volts, having a frequency of from 40 to 100 periods per second, is applied for 10 minutes between the conductor and the water, the test piece previous to immersion having been bent six times (three times in one direction and three times in the opposite direction) round a smooth cylindrical surface not more than twelve times the diameter of the conductor measured outside the dielectric.

The coil from which the test piece was cut should be tested in a similar manner to class A, but the minimum insulation resistance should be 300 megohms per mile for all sizes of conductors. Conductors of class A must be protected from mechanical injury by being covered with stout braid or taping, prepared so as to resist moisture, and must be further protected by casing, or by being drawn into pipes or conduits. In the case of conductors insulated as in class B, great care must be taken to protect exposed ends of conductors where they enter the terminals of switches, fuses, and other appliances, from the possible access of moisture which might creep along the insulating material within the water-proof covering.

Worked Examples. (1) A supply company insists that the insulation resistance of an installation shall not be less than 0·5 megohm. Determine (1) the insulation resistance of the system if there are 250 consumers, and (2) the total leakage current if the supply voltage is 200 volts.

(1) Since the insulation resistance of the system is the joint insulation resistance of 250 branches connected in parallel, the insulation resistance of the system is

$$R_i = \frac{0.5\ \Omega}{250} = \frac{500000\ \omega}{250}$$
$$= 2000 \text{ ohms.}$$

(2) Total leakage current
$$= \frac{200}{2000}$$
$$= 0{\cdot}1 \text{ ampere.}$$

(2) The conductor resistance of a well-insulated telegraph line whose farther end is to earth is 4000 ohms; if a fault exists at its centre of

1000 ohms resistance, determine (1) the insulation resistance of the circuit with the fault, and (2) the current which leaks to earth at the centre, and the amount which is received at the far end, if a battery of 5 volts and of negligible resistance is introduced at the other end.

(1) Using the symbols given above

$$r_2 = r_a + \frac{f \times r_b}{f + r_b} \; (\beta)$$

it is obvious that $\quad r_a = \dfrac{4000}{2} = 2000$ ohms, and $r_b = 2000$ ohms

$$\therefore \quad r_2 = 2000 + \frac{1000 \times 2000}{1000+2000}$$
$$= 2666\tfrac{2}{3} \text{ ohms.}$$

(2) Current traversing the battery is

$$C = \frac{5}{2666\frac{2}{3}} = \frac{3}{1600} \text{ ampere.}$$

This current divides at the point B, so that the leakage current, c_1, is

$$c_1 = \frac{r_b}{f + r_b} \times C = \frac{2000}{1000+2000} \times \frac{3}{1600}$$
$$= \frac{1}{800} \text{ ampere.}$$

And the current received at the far end is

$$c_2 = \frac{f}{f + r_b} \times C = \frac{1000}{1000+2000} \times \frac{3}{1600}$$
$$= \frac{1}{1600} \text{ ampere.}$$

(3) The insulation resistance of a mile of insulated cable is 300 megohms, and the diameter of the conductor is 2 millimetres, the thickness of the insulating envelope being 1 millimetre. Determine the thickness of the envelope of the same material for a mile of cable, if the diameter of the conductor is 3 millimetres and the insulation resistance of the cable be 600 megohms.

Since $\quad R_i = \dfrac{K}{l} \log \dfrac{D}{d}$

we have
$$\left.\begin{aligned} 300 &= \frac{K}{l} \log \frac{4}{2} \\ 600 &= \frac{K}{l} \log \frac{D}{3} \end{aligned}\right\}$$

where D = outside diameter of the new covering

and
$$\frac{600}{300} = \frac{\log \frac{D}{3}}{\log \frac{4}{2}} = \frac{\log \frac{D}{3}}{\log 2}$$

$$\therefore \quad \log \frac{D}{3} = 2 \log 2$$
$$= \log 2^2 = \log 4$$

and $\quad \dfrac{D}{3} = 4$ or $D = 12$

but $\quad D = 3 + 2t$

where $\quad t$ = thickness of insulation

$$\therefore \quad t = \frac{12-3}{2} = 4\tfrac{1}{2} \text{ millimetres.}$$

(4) A certain cable is 1 kilometre in length, and the core is 2·5 millimetres in diameter. If the dielectric is 1·25 millimetres thick and its resistivity $4{\cdot}5 \times 10^{14}$ ohms, determine the insulation resistance of the cable.

Since $$R_i = \frac{\sigma}{2{\cdot}728\, l} \log \frac{D}{d}$$

and $$\sigma = 4{\cdot}5 \times 10^{14} \text{ per cubic cm.}$$
$$l = 1000 \times 100 = 100000 \text{ cms.}$$

and $$R_i = \frac{4{\cdot}5 \times 10^{14}}{2{\cdot}728 \times 100000} \log \frac{5{\cdot}0}{2{\cdot}5}$$
$$= \frac{45 \times 10^{11}}{2728} \log 2$$

and $$\log 2 = 0{\cdot}30103$$
$$\therefore \quad R_i = \frac{45 \times 10^{11} \times {\cdot}30103}{2728} \text{ ohms}$$
$$= 497{\cdot}3 \text{ megohms.}$$

EXERCISES II c.

(1) The total insulation of a telegraph line between terminal stations A and C is 5000 ohms, and the total insulation between terminal station A and intermediate station B is 7000 ohms. What is the total insulation of the section B to C? (C. and G.)

(2) The total insulation resistance of a telegraph line 100 miles long is found to be 5000 ohms, and the total insulation resistance of the line up to a point 60 miles from where the measurement is made is found to be 8000 ohms. What is the insulation resistance per mile of the remaining 40 miles of line? (C. and G.)

(3) A distributing circuit is 1000 yards in length and consists of cable, the insulation resistance of which is 500 megohms per mile. Determine the leakage current if the supply voltage is 220 volts and there are 25 joints of 20000 megohms insulation resistance and 300 fittings of 900 megohms each.

(4) The conductor resistance of a well insulated telegraph line whose farther end is to earth is 2000 ohms; if there were a fault at its centre of 1000 ohms resistance, what would the resistance of the line be reduced to? (C. and G.)

(5) The conductor resistance of a well insulated telegraph line 100 miles long is 1250 ohms. The line breaks at a certain point making a dead earth, and the resistance is then found to be 250 ohms, determine the distance of the fault from the testing station.

(6) The conductor resistance of a perfect line 100 miles long is 1800 ohms. A fault is introduced at an intermediate point, and with the far end insulated the resistance is then 1400 ohms. But when the far end is earthed the resistance becomes 1080 ohms, determine the position of the fault and its resistance.

(7) A cable whose conductor resistance when perfect is 1000 ohms becomes faulty, and the resistance measured from the nearer end when the further end is insulated is found to be 900 ohms; when the further end is put to earth then the resistance measured from the nearer end is found to be 820 ohms. Determine the position of the fault. (C. and G.)

(8) What is the resistance of the fault in No. 7?

(9) The strength of current sent in at one end of a telegraph line is 100 milliamperes, and the current received at the other end is 30 milliamperes; there is known to be a fault at the centre of the line, and the resistance of the line wire is 2000 ohms. What is the resistance of the fault? (C. and G.)

(10) A telegraph line, 300 miles long, and having a conductor resistance of 10 ohms per mile, has a fault at its centre of 5000 ohms resistance. What will be the strength of current sent out from a 50-cell Daniell battery (E.M.F. per cell, 1·07 volt), and the strength of current received at the farther end on a galvanometer of negligible resistance? (C. and G.)

(11) If three lengths of cable, having respectively an insulation resistance of 300, 400, and 500 megohms, be joined in a continuous length, what will be the insulation resistance of the whole cable? (C. and G.)

(12) The insulation resistance of a mile of insulated cable is 300 megohms, and the diameter of the conductor is ·2 millimetres, the thickness of the insulating covering being 1 millimetre. Show that if we want to have an insulation resistance of 600 megohms, the thickness of the insulating material must be increased to 3 millimetres.

(13) Two submarine cables of equal length have conductors whose diameters are respectively 80 and 100 mils, the thickness of their guttapercha coverings being 120 and 80 mils. Determine the ratio of their insulation resistances. Given $\log 2 = \cdot 30103$; $\log 13 = 1\cdot 1139$.

(14) A certain submarine cable is 525 miles long; the core is 2·8 millimetres in diameter, and the outside covering of the insulation is 7·85 millimetres. The resistivity of the dielectric is $1\cdot 5 \times 10^{17}$ megohms (per cm. cube). Determine its insulation resistance per mile, and the total insulation resistance of the cable.

§ **24. Capacity of Conductors and Condensers.** The practical unit of current, the ampere, is defined as the rate of flow or transference of unity quantity (the coulomb) of electricity per second, so that in a continuous-current circuit a charge (or quantity) of electricity equal to Ct or Q coulombs passes any one point in the circuit in the

time t if a current of C amperes traverse the circuit for t seconds, consequently the rate of flow is

$$C = \frac{Q}{t} = \frac{\delta Q}{\delta t}.$$

Now if the circuit be broken at any point the current ceases, and if two adjacent and separate conductors or metal plates, well insulated from one another, are connected respectively to the two free ends, we shall have the arrangement shown in Fig. 13, and since the two insulated conductors are connected to the terminals of a battery,

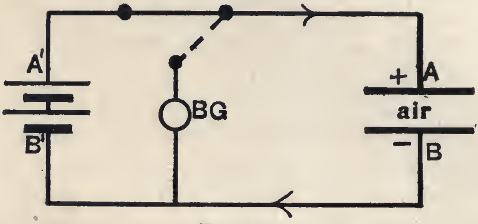


FIG. 13.

a transient rush of electricity takes place, as indicated by the arrows, and each plate receives an electrostatic charge Q (say). The plate A is electrified positively and B negatively. Let V_A and V_B be the potentials of the plates A and B, such that $V_A > V_B$, then

$$V_A - V_B = \text{P.D. between the plates.}$$

Obviously the plates A and B are electrified to the same P.D. as the poles A′ and B′ of the battery, and

$$Q \propto (V_A - V_B)$$

or

$$Q = K(V_A - V_B).$$

The constant of proportionality K, which represents the ratio of the charge Q on each plate to the P.D. $(V_A - V_B)$, is termed the capacity of the arrangement. If $(V_A - V_B) = 1$, when the charge is q units—i. e. if q units of electricity charge the system to unit difference of potential—we have

$$q = K$$

from which the following definition is derived :—*The C.G.S. unit of capacity is measured by the number of C.G.S. units of electricity required to raise its potential to* 1 *C.G.S. unit.* In the practical system of units, the unit of capacity is the *farad*, which is the capacity of conductor which requires one coulomb to charge it to a difference of potential of 1 volt.

It may be proved experimentally that the capacity of two insulated conductors is proportional to the area of the surfaces exposed to one another, and inversely proportional to the distance between them; but the nature of the insulating material separating them also influences the capacity to a large extent, and the capacity is directly

proportional to the *dielectric constant* or *specific inductive capacity*, k, of the insulating medium. It thus follows that the capacity of a single transmission line will be increased by bringing near to it another conductor, but separated from it by some good insulating material or air, and any device for increasing the capacity of a conductor in this way is termed a *condenser*.

Long distance lines, underground cables, especially concentric cables, and insulated cables generally, possess considerable capacity, and in many cases the capacity of a circuit is an important factor in practical work. Standard condensers (see Fig. 14) usually consist of two sets of parallel plates of tinfoil or other thin conducting sheets, separated by very thin sheets of mica, paraffined paper, or other insulating sheets. The conducting plates are alternately connected together, so as to form practically two plates of large area, and the thickness of the dielectric is very small compared with the dimensions

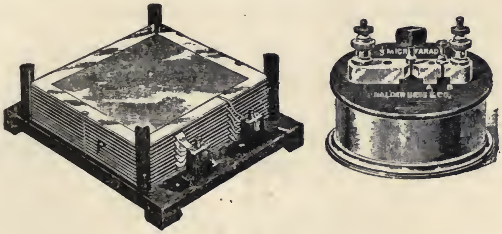


FIG. 14.

of the plates. The conducting plates are usually connected to two brass pieces, poles or terminals. When a condenser is charged, one plate receives a quantity of positive electricity, and the other an equal quantity of negative electricity.

For practical purposes the farad is too large a unit, since the capacity of the earth is only about 0·0007 farad, and the one-millionth part of a farad, i. e. 1 *microfarad*, is usually taken as the working practical unit of capacity. A condenser with a capacity of 1 microfarad thus requires only one-millionth of a coulomb to produce a difference of potential between the plates of 1 volt. Standard condensers of one-third microfarad capacity are much used for testing purposes, since the value is approximately equal to the capacity of one nautical mile of the Atlantic cable.

Since $Q = K V$

$$K \text{ (farads)} = \frac{Q \text{ (coulombs)}}{V \text{ (volts)}}$$

$$= \frac{\text{charge on one plate}}{\text{P.D. between the plates.}} = \text{a ratio}$$

$$\therefore \quad 1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{\frac{1}{10}\text{C.G.S. unit of quantity}}{10^8 \text{ C.G.S. units of pressure}}$$

$$= \frac{1}{10^9}\text{C.G.S. unit of capacity}$$

but one microfarad $= \frac{1}{10^6}$ of a farad

$$= \frac{1}{10^9 \times 10^6} \text{ or } \frac{1}{10^{15}} \text{ C.G.S. unit of capacity.}$$

The C.G.S. unit here referred to is the absolute electromagnetic unit, and it may be shown that the microfarad is approximately 900000 times greater than the absolute electrostatic unit of capacity (see Chapter XI).

The following formulae enable the capacities of different condensers to be readily calculated. Thus the capacity of an air condenser with parallel plates is

$$K_e \text{ (electrostatic units)} = \frac{A \text{ (sq. cms.)}}{4\pi t \text{ (cms.)}}$$

where t = distance between the plates, or the thickness of the dielectric, and if K_{mf} = capacity in microfarads

$$\left.\begin{aligned} K_{mf} &= \frac{A \text{ (sq. cms.)}}{4\pi \times 900000\, t \text{ (cms.)}} \\ &= \frac{6{\cdot}54 \text{ A (sq. in.)}}{2{\cdot}54 \times 4\pi \times 900000\, t \text{ (in.)}} \\ &= \frac{A \text{ (sq. in.)}}{4{\cdot}4526 \times 10^6 t \text{ (in.)}} = \frac{2{\cdot}25 \text{ A (sq. in.)}}{10^7\, t \text{ (in.)}} \end{aligned}\right\} \quad . \; . \; . \; . \quad (44)$$

If the insulating medium or dielectric has a specific inductive capacity k, the above formulae must be multiplied by the particular value of k for the material used.

The specific inductive capacity of

air	= 1	indiarubber	= 2·5
glass	= 3 to 3·25	paraffin	= 2·0
ebonite	= 2·284	sulphur	= 2·58
guttapercha	= 2·462	mica	= 5·0

The capacity of concentric cylindrical condensers—of which concentric and submarine cables are examples—is

$$K_e = k \times \frac{l \text{ (cms.)}}{2 \log_\epsilon \frac{r_1}{r}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

where l = length, and r_1, r = the external and internal radii or diameters respectively

$\epsilon = 2{\cdot}71828$ the base of the Napierian logarithms

$$\therefore \quad K_e = k \times \frac{l\,(\text{cms.})}{2\log_{10}\frac{r_1}{r} \times 2{\cdot}3026} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and

$$\left.\begin{aligned} K_{mf} &= k \times \frac{l\,(\text{cms.})}{2 \times 2{\cdot}3026 \times 900000 \times \log_{10}\frac{r_1}{r}} \\ &= k \times \frac{2{\cdot}413}{10^7} \times \frac{l\,(\text{cms.})}{\log_{10}\frac{r_1}{r}} \\ &= k \times \frac{12 \times 2{\cdot}54\, l\,(\text{ft.})}{2 \times 2{\cdot}3026 \times 900000 \times \log_{10}\frac{r_1}{r}} \end{aligned}\right\} \quad . \quad . \quad . \quad (47)$$

$$= k \times \frac{7{\cdot}354}{10^6} \times \frac{l\,(\text{ft.})}{\log_{10}\frac{r_1}{r}}$$

The following values are given as the capacity between the inner and outer conductors of high tension electric light cable (19/18 concentric cable) per mile.

British Insulated Wire Company (paper) .	0·31 microfarad.
W. T. Glover & Co. (vulcanised rubber) . .	0·615 microfarad.
W. T. Glover & Co. (diatrine)	0·315 microfarad.
Ferranti mains (compressed brown paper and black wax)	0·3675 microfarad.

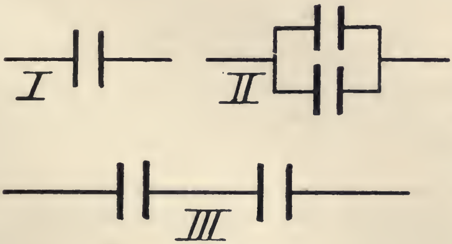


FIG. 15.

Condensers are represented symbolically, as shown in Fig. 15 I, and we may note in passing that condensers may be connected in *parallel* and in *series*. When connected in parallel (Fig. 15 II) it is clear that the combination simply enlarges the areas of the plates,

from which it follows that the joint capacity is equal to the sum of the several capacities; or

$$K = k_1 + k_2 + k_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Connecting condensers in series (Fig. 15 III) is equivalent to increasing the thickness of the dielectric, from which it follows that the joint capacity will be less than the capacity of any one of the several condensers; in fact the reciprocal of the joint capacity of condensers in series is equal to the sum of the reciprocals of the several capacities, or

$$\frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

and

$$K = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Worked Examples. (1) Three condensers of the respective capacities 1, 2, and 3 microfarads are joined up in 'cascade'; what is the capacity of the combination?

$$\frac{1}{K} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

$$= \frac{6+3+2}{6} = \frac{11}{6}$$

$$\therefore \quad K = \frac{6}{11} \text{ microfarad.}$$

(2) A condenser of 30 microfarads capacity is connected in cascade with a second condenser, and it is found that the capacity of the combination is 12 microfarads; what is the capacity of the second condenser?

Since

$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\frac{1}{K_2} = \frac{1}{K} - \frac{1}{K_1} = \frac{1}{12} - \frac{1}{30}$$

$$= \frac{5-2}{60} = \frac{3}{60} = \frac{1}{20}$$

$$\therefore \quad K_2 = 20 \text{ microfarads.}$$

(3) A condenser of 10 microfarads capacity is fully charged, and then connected to a second condenser of unknown capacity; it is found that two-thirds of the charge has been lost; what is the capacity of the second condenser?

The charge given to the first condenser will distribute itself between the two condensers proportionately to their relative capacities.

Let K_1 and K_2 be the respective capacities of the two condensers and Q the charge on the first when fully charged and not connected with second. Also let Q_1 and Q_2 be the charges on the two condensers respectively when connected.

Then

$$Q_1 = Q \times \frac{K_1}{K_1 + K_2}$$

$$\therefore \quad Q_1 (K_1 + K_2) = QK_1$$

and

$$Q_1 K_2 = QK_1 - Q_1 K_1$$

$$\therefore \quad K_2 = \frac{K_1\,(Q - Q_1)}{Q_1}$$

but $$Q_1 = \tfrac{1}{3}\,Q \text{ and } Q_2 = \tfrac{2}{3}\,Q.$$

Substituting these values

$$K_2 = \frac{10\,(Q - \frac{1}{3}\,Q)}{\frac{1}{3}\,Q}$$
$$= 10 \times 2 = 20 \text{ microfarads.}$$

(4) A copper wire ·02 inch in diameter is insulated by a covering of guttapercha ·01 inch in thickness; what thickness of guttapercha would be required to insulate a copper wire ·03 inch in diameter in order that its inductive capacity may be the same as that of the first wire?

Let d_1 and D_1 be the external diameters, and d and D the internal diameters of the two wires, then

$$d = \cdot 02,\; d_1 = \cdot 02 + 2 \times \cdot 01 = \cdot 04$$
$$D = \cdot 03,\; D_1 = \cdot 03 + 2\,t$$

where t = the thickness of the guttapercha required;

then $$\frac{K_1}{K_2} = \frac{\log \dfrac{d_1}{d}}{\log \dfrac{D_1}{D}}.$$

but $$K_1 = K_2$$

$$\therefore \quad \log \frac{\cdot 04}{\cdot 02} = \log \frac{(\cdot 03 + 2\,t)}{\cdot 03}$$

$$\therefore \quad \log 2 = \log \frac{(\cdot 03 + 2\,t)}{\cdot 03}$$

and $$\frac{\cdot 03 + 2\,t}{\cdot 03} = 2$$

$$\therefore \quad \cdot 03 + 2\,t = 2 \times \cdot 03 = \cdot 06$$

$$\therefore \quad 2\,t = \cdot 06 - \cdot 03 = \cdot 03$$

and $$t = \cdot 015 \text{ inch.}$$

(5) Determine the capacity of a mile of the concentric cable forming the trunk main connecting Deptford to the substations in the Metropolis. A section of the cable is given in Fig. 16.

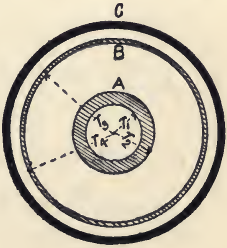


FIG. 16.

The inner tube A, and the outer tube B, are made of copper, whilst C is a tube of sheet-iron enclosing the cable, the three tubes being insulated from one another by an insulating material made of compressed brown paper and black wax, the specific inductive capacity of the whole being 3·37.

The values of r_1, r_2, r_3, and r_4 in the figure are as follows:

$r_1 = \cdot 281$ inch
$r_2 = \cdot 406$ inch
$r_3 = \cdot 922$ inch
$r_4 = \cdot 959$ inch

from which the ratio $\frac{r_1}{r}$ in the formula given above is

$$\frac{r_1}{r} = \frac{\cdot 922}{\cdot 406} = 2\cdot 27$$

$$\therefore \quad \log_e \frac{r_1}{r} = \log_e \frac{\cdot 922}{\cdot 406} = \{\log_{10}(2\cdot 27)\} \times 2\cdot 3026$$

and since $$l = 1 \text{ mile} = (5280 \times 12 \times 2\cdot 54) \text{ centimetres}$$

∴ the capacity per mile in microfarads is

$$K = \frac{3{\cdot}37 \times 5280 \times 12 \times 2{\cdot}54}{2\,(\log 2{\cdot}27) \times 2{\cdot}3026 \times 900000}$$
$$= 0{\cdot}3675 \text{ microfarad.}$$

Another formula for calculating the capacity of a concentric cable in microfarads is given by Ayrton as follows:

$$K = k\frac{2{\cdot}413}{10^7} \times \frac{l\ (\text{cms.})}{\log_{10}\frac{r_1}{r}}$$

which may be shown to be identical with the formula used.

(6) One of the Ferranti mains between Deptford and London has a capacity of 3 microfarads. What is the work stored in it, in ergs, if the inner and outer conductors are charged to a potential difference of 10000 volts?

When a condenser is charged to a certain difference of potential E, the store of electrical energy in it or the work which it can do when discharged is given by the formula

$$W\ (\text{ergs}) = \frac{K \times E^2}{2}$$

where K and E are the values of the capacity and P.D. in C.G.S. units respectively.

$$\therefore \quad W = \tfrac{1}{2} \times 3 \text{ microfarads} \times (10000 \text{ volts})^2$$

and
$$W \text{ ergs} = \tfrac{1}{2} \times \frac{3}{10^{15}} \times (10000 \times 10^8)^2$$
$$= \tfrac{1}{2} \times \frac{3}{10^{15}}(10^{12})^2$$
$$= \frac{3 \times 10^{24}}{2 \times 10^{15}} = \frac{3}{2} \times 10^9$$
$$= 1{\cdot}5 \times 10^9 \text{ ergs.}$$

Note.—Since $1 \text{ farad} = \frac{1}{10^9}$ of the absolute unit of the capacity

$1 \text{ microfarad} = \frac{1}{10^{15}}$ of the absolute unit of the capacity.

To get the work stored in foot-pounds, Ayrton gives this relationship:

$$W\ (\text{foot-pounds}) = \frac{K\ (\text{farads}) \times E^2\ (\text{volts})}{2{\cdot}712}$$

∴ substituting the values given we get

$$W\ (\text{foot-pounds}) = \frac{3}{10^6} \times \frac{(10000)^2}{2{\cdot}712}$$
$$= \frac{300}{2{\cdot}712} = \frac{100}{{\cdot}904} = 110{\cdot}6.$$

As an example showing how to change from foot-pounds to ergs, the following addition will be useful.

Since $1 \text{ foot-pound} = 13540000 \text{ ergs}$

$$\therefore \quad W \text{ ergs} = \frac{100}{{\cdot}904} \times 13540000$$
$$= \frac{1354}{904} \times 10^9 \text{ ergs}$$
$$= 1{\cdot}5 \times 10^9 \text{ ergs.}$$

EXERCISES II D.

(1) Calculate the capacity in microfarads of a condenser composed of two plates, each $1\frac{1}{2}$ metres long, 65 cms. wide, placed parallel to one another at a distance of 0·2 mm. apart. 1 farad equals 9×10^{11} C.G.S. electrostatic units of capacity. (C. and G.)

(2) Calculate the size of plates that must be separated by a layer of air 0·01 inch in thickness, so that the air condenser may have 0·0001 of a microfarad capacity. (C. and G.)

(3) Determine the quantity of electricity which would exist on 20 square centimetres area of a plate at zero potential, when separated by 0·43 mm. from a plate charged to a potential of 6 C.G.S. units.

(4) The capacities of two Leyden jars are as 3 to 5, and they have charges which are as 2 : 3. Compare the quantities of heat produced by discharging them.

(5) A condenser is made of 100 pairs of square plates 20 cm. side, 0·5 mm. apart, immersed in oil of specific inductive capacity 2. Determine the capacity in microfarads.

(6) How would you combine four condensers, each having a capacity of 1 microfarad, so as to produce a capacity of 0·75 microfarad? (C. and G.)

(7) Two submarine cables of equal length have conductors whose diameters are 80 and 100 mils, the diameters of the guttapercha coverings being 120 and 80 mils. Determine the relative capacities of the two cables. (C. and G.)

$$\log 2 = \cdot 30103 \qquad \log 3 = \cdot 47712.$$

(8) The capacity of 1 mile of copper wire 50 mils in diameter, covered with guttapercha to a thickness of $62\frac{1}{2}$ mils, is 0·29 microfarad; what is the capacity of a wire 100 mils diameter covered to a diameter of 390 mils? (C. and G.)

$$\log 3{\cdot}5 = \cdot 5440680 \qquad \log 3{\cdot}9 = \cdot 5910646.$$

§ **25. Arrangement of Cells.** Voltaic cells may be connected or grouped together in three different ways to form batteries, as shown in Fig. 17 I, II and III. When a series of voltaic cells are so grouped that the negative pole of one is connected to the positive pole of the next (Fig. 17 I), the cells are said to be connected *in series*, and to determine the electromotive-force of a battery with cells connected in series we have only to add together the electromotive forces of the individual cells. Thus, if n cells of E.M.F. E volts are connected in series, the E.M.F. of the battery is nE volts. Similarly, the internal resistance of a battery with cells connected in series is the

sum of the individual internal resistances of the cells, and if r_b is the internal resistance of n equal and similar cells, the total internal resistance of the battery is nr_b ohms. If a battery of n cells is closed by an external resistance R, we have, by Ohm's Law, for the resulting current

$$C = \frac{nE}{R + nr_b} \quad \ldots \ldots \ldots \quad (50)$$

If, however, all the positive poles of n similar cells are connected to one terminal of the circuit, and all the negative poles to another terminal, as shown in Fig. 17 II, the cells are said to be connected *in parallel*. It is clear that with this arrangement the battery is

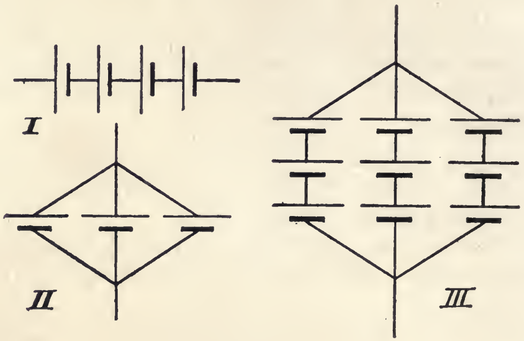


FIG. 17.

equivalent to a single large cell whose E.M.F. is equal to that of each of the n cells, the plates of which are n times as large as those of the single cells, then if the plates are the same distance apart in both cases, the internal resistance of the battery is $\frac{1}{n}$th of r_b or $\frac{r_b}{n}$. The current supplied to an external circuit of R ohms resistance is therefore

$$C = \frac{E}{R + \frac{r_b}{n}} = \frac{nE}{nR + r_b} \quad \ldots \ldots \ldots \quad (51)$$

Now, to produce a required effect by means of a battery in the best possible manner—i. e. to get the greatest current through a given external resistance, R—it is often found that cells grouped so as to form a combination of the series and parallel arrangements, as shown in Fig. 17 III, give the best results. Thus, with N equal cells, we may divide them into m sets consisting of n cells connected in series, and by connecting the m sets in parallel form a compound arrangement. The E.M.F. of each row or set is obviously nE, and the internal resistance of each row, nr_b, and since the m rows are

connected in parallel the E.M.F. of the combination is nE and its internal resistance $\frac{n}{m} r_b$. The current through an external resistance R ohms is, therefore,

$$C = \frac{nE}{R + \frac{n}{m} r_b} = \frac{E}{\frac{R}{n} + \frac{r_b}{m}}. \quad \quad (52)$$

To decide which is the best arrangement for connecting voltaic cells, we may deduce from the results given by the equations (50), (51), and (52), some general principles which will enable us to determine which arrangement will give the greatest current or best efficiency according to the conditions imposed. Whether any advantage is to be gained by connecting cells in series or in parallel, or by using more than one cell, depends entirely upon the relative values of R and r_b. If R is great compared with r_b, then, as equation (43) indicates, the current is increased by increasing the number of cells connected in series, but by (44) very little is to be gained by increasing the number of cells connected in parallel. On the other hand, if R is small compared with r_b there is very little or no gain by increasing the number of cells connected in series, but a distinct increase in the current by increasing the number connected in parallel. For this reason cells connected in parallel are sometimes said to be connected for quantity. If it is desired to get the greatest current possible through a given external resistance by means of a given number of cells, the best arrangement of grouping is that which makes the total internal resistance as nearly as possible equal to the external resistance. The arrangement which satisfies this condition best, however, is not economical, since there is as much energy wasted in the battery as heat as there is energy utilized externally, and the efficiency is only 50 per cent. From equation (45) it is clear that C is a maximum when the arrangement of grouping makes the denominator $\left(\frac{R}{n} + \frac{r_b}{m}\right)$ a minimum for given values of N, r_b, and R.

Now,

$$\left(\frac{R}{n} + \frac{r_b}{m}\right)^2 = \frac{R^2}{n^2} + \frac{2\,Rr_b}{nm} + \frac{r_b^2}{m^2}$$

$$= \frac{R^2}{n^2} - \frac{2\,Rr_b}{nm} + \frac{r_b^2}{m^2} + \frac{4\,Rr_b}{nm}$$

$$= \left(\frac{R}{n} - \frac{r_b}{m}\right)^2 + \frac{4\,Rr_b}{nm}.$$

But $\frac{4\,Rr_b}{nm}$ or $\frac{4\,Rr_b}{N}$ is constant for all values of n and m, conse-

quently $\left(\frac{R}{n} + \frac{r_b}{m}\right)^2$ is a minimum when $\left(\frac{R}{n} - \frac{r_b}{m}\right)$ is zero. In other words, the denominator $\left(\frac{R}{n} + \frac{r_b}{m}\right)$ in equation (45) is as small as possible when

$$\frac{R}{n} - \frac{r_b}{m} = 0 \quad \text{or when} \quad \frac{R}{n} = \frac{r_b}{m}$$

$$\text{i. e. when } R = \frac{n r_b}{m}.$$

But $\frac{n r_b}{m}$ is the internal resistance of the battery with the combination or multiple-series arrangement, so that when the internal resistance of the battery is equal to the external resistance the resulting current is a maximum.

Again, when it is desired to obtain the same current from a given number of cells through an external resistance, whether connected in series or in parallel, the following relation must exist, i. e. r_b must equal R. Thus

$$C_s = \frac{nE}{R + n r_b} \quad \text{and} \quad C_p = \frac{nE}{nR + r_b}$$

and if $$C_s = C_p,$$
then $$R + n r_b = nR + r_b$$
and $$(n-1) r_b = (n-1) R$$
or $$r_b = R.$$

When the current through an external resistance, R, is a maximum, the terminal P.D. of the battery will be half of the E.M.F. of the battery. If E = E.M.F. of the battery, and r_b' = internal resistance of the battery

$$E = C(R + r_b')$$

but when C is a maximum $R = r_b'$

$$\therefore \quad E = 2CR$$

But $CR = e$, the terminal P.D. of the battery

$$\therefore \quad E = 2e.$$

To determine the number of cells connected in series required to give a certain current through a known resistance apply the following rule. 'Divide the P.D. required externally by the E.M.F. of a single cell diminished by the volts lost internally in each cell.' Let n = the number of cells connected in series, then

$$C = \frac{nE}{n r_b + R}$$

and $$nE = C(n r_b + R)$$

or $$n = \frac{CR}{E - C r_b};$$

but $$C R = e = \text{terminal P.D.}$$

$$\therefore \qquad n = \frac{e}{E - C r_b}.$$

The best working arrangement of cells is that which gives the best efficiency, and efficiency is defined as the ratio of the output to the input, or

$$\text{efficiency} = \eta = \frac{\text{energy used externally}}{\text{total energy developed}}$$

$$= \frac{R}{R + r_b'}$$

$$\therefore \qquad \left.\begin{matrix}\text{total resistance}\\ \text{of circuit}\end{matrix}\right\} = \frac{R}{\eta} = R + r_b'$$

$$\text{Internal resistance } r_b' = \frac{R(1-\eta)}{\eta}.$$

Therefore, to determine the E.M.F., E′, of a battery required to send a given current through a known external resistance with a given efficiency, we have

$$\frac{E'}{C} = R + r_b' = \frac{R}{\eta}$$

$$\therefore \qquad E' = \frac{R}{\eta} \times C.$$

The activity of a cell when short-circuited expressed in watts is of the nature of a constant, which gives the *electrical capability* of the cell. Since the short-circuited current is $\frac{E}{r_b'}$ or C_{sh}, the *electrical capability* is $C_{sh}E$ or $\frac{E^2}{r_b'}$, consequently when it is required to determine the minimum number of cells which will supply an activity of P watts to an external circuit, the current required will be the maximum value, in which case, as we have already shown, only half of the pressure of the battery will be available for external use, and the minimum number of cells is

$$n_1 = \frac{P}{\left(\frac{E}{2}\right)^2 \div r_b'} = \frac{4P}{\frac{E^2}{r_b'}}.$$

With this number of cells the first cost is a minimum, and the efficiency is only 50 per cent.; it may also be shown that the method of grouping the cells does not affect their activity. It does not follow, however, that a battery designed for minimum installation cost will be able in all cases to maintain the theoretical short-circuit current and the desired activity on account of polarization, &c. If C_m is the maximum current the cell can sustain, the activity supplied

to the external circuit per cell will be $EC_m - C_m^2 r_b$ watts, consequently the minimum number of cells required for a useful activity P is given by

$$n_2 = \frac{P}{EC_m - C_m^2 r_b}.$$

The efficiency in this case is

$$\eta = \frac{EC_m - C_m^2 r_b}{EC_m}$$
$$= 1 - \frac{C_m r_b}{E}$$

and since C_m is less than the current corresponding to the minimum installation cost value $\frac{C_m r_b}{E}$ is less than 0·5, and, therefore, η is greater than 0·5 or 50 per cent.

Universal Battery System. The universal battery system is a method of grouping several circuits (usually five) at a telegraph

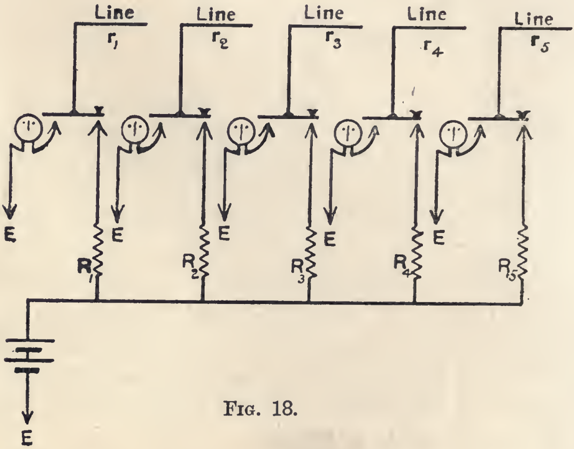


FIG. 18.

station to one battery, which permits the independent working of any number of these circuits at the same time. The theory of this system will be understood by considering the following example.

In Fig. 18 let a battery of n cells be connected to five circuits grouped in parallel to the same terminal of the battery, and to simplify matters let us assume that the resistances of the lines, r_1, r_2, r_3, r_4, and r_5 be the same and equal to r, and also let the resistances in the battery leads—i.e. R_1, R_2, R_3, R_4, and R_5—be considered equal to one another, then if the E.M.F. of the battery be E, and the

internal resistance of the battery negligibly small, the current in any one circuit working singly will be

$$c = \frac{E}{R} \text{ amperes};$$

if R = total resistance of each circuit = $r_1 + R_1$, whilst, if the five circuits are in use at the same time, the total current passing through the five circuits will be

$$C = \frac{E}{\frac{1}{5}R} = \frac{5\,E}{R} \text{ amperes},$$

since the external circuits are connected in parallel; again, since each circuit has the same resistance, this current will divide equally between them, and the current in each branch will be

$$c = \tfrac{1}{5}\,C = \frac{E}{R} \text{ amperes},$$

that is, there is no difference (theoretically) whether one or any number of the circuits are in use provided the battery resistance be zero.

In practice, however, this does not hold, inasmuch as the battery resistance affects the available P.D. at the battery terminals as the number of circuits is varied. In any case, if e be the P.D. at the battery terminals,

$$e = \frac{\frac{R}{m}}{r_b' + \frac{R}{m}} \times E = \frac{R}{m\,r_b' + R} \times E$$

where m = number of circuits connected in parallel, from which it is evident that the current will vary with the number of circuits in use, unless r_b' be very small compared with R. In fact, the number of circuits which may be worked independently from one battery by this system is limited by the internal resistance of the battery. Moreover, it is not probable that in practice the resistance of the circuits will be the same, and experience shows that not more than 25 per cent. variation between the highest and lowest resistances must exist if this system be adopted. If one or more circuits have a resistance less than 75 per cent. of the highest resistance, that is, less than $\left(r_1 - \frac{25}{100} r_1\right)$ if r_1 be the highest resistance, an equalizing coil (R_1, R_2, &c., in the figure) is connected in the battery lead of the circuit to bring it up to the required value. It is also important to notice that the internal resistance of the battery must in no case exceed that of the joint resistance of the grouped circuits.

To determine the battery power required to work a group of circuits on the universal battery system proceed as follows:—

Let C = the total current required;
E = the E.M.F. of each cell;
n = the number of cells required;
r_b = the internal resistance of each cell;
R = the joint resistance of the circuits grouped in parallel;

then
$$C = \frac{nE}{nr_b + R}$$
and
$$nr_bC + CR = nE$$
and
$$n(E - Cr_b) = CR$$
$$\therefore \quad n = \frac{CR}{E - Cr_b} = \frac{R}{\frac{E}{C} - r_b}$$
or in words

Number of cells required
$$= \frac{\text{External resistance}}{\frac{\text{E.M.F. of one cell}}{\text{current required}} - \text{internal battery of one cell.}}$$

The determination of the current in any branch is an easy matter when the resistance of each circuit is the same; for instance, with m circuits grouped together all working at once the total current is
$$C = \frac{E}{r_b' + \frac{R}{m}}$$
$\therefore$ current in each branch is
$$c = \frac{1}{m}\left\{\frac{E}{r_b' + \frac{R}{m}}\right\} = \frac{E}{mr_b' + R}.$$
This formula holds whether m is 1, 2, 3, 4 or 5.

Worked Examples. (1) How many cells of 0·75 ohm internal resistance connected in series, will be required to send half an ampere through each of two wires 50 ohms resistance connected in parallel between the terminals of the battery, if twenty-five of these cells send half an ampere through an external resistance of 50 ohms?

Let E = E.M.F. of each cell
n = number of cells connected in series

then with the example given
$$0{\cdot}5 = \frac{25 \times E}{50 + (25 \times {\cdot}75)} = \frac{E}{2 + {\cdot}75}$$
$$\therefore \quad E = 0{\cdot}5 \times 2{\cdot}75 = 1{\cdot}375 \text{ volts.}$$
Now, since each conductor carries half an ampere, the current supplied is 1 ampere, and the joint resistance of the two conductors is $\frac{50}{2}$, we have
$$1 = \frac{n \times 1{\cdot}375}{\frac{50}{2} + n \times 0{\cdot}75}$$

$$\therefore \quad 25+\tfrac{3}{4}n = 1\tfrac{3}{8}n$$

or $$\tfrac{5}{8}n = 25 \text{ and } n = 40 \text{ cells.}$$

(2) Determine the greatest current which can be obtained from forty cells of E.M.F. 1·4 volts and 4 ohms internal resistance, if the external resistance is 40 ohms.

Since the current is a maximum when the internal resistance of the battery is as nearly equal to the external resistance as possible, then if there are m rows connected in parallel, the number of cells in series will be $\frac{40}{m}$, and the internal resistance is $\frac{40}{m^2} \times 4$. To find m we have

$$\frac{40}{m^2} \times 4 = 40$$

or $$m^2 = 4$$

and $$m = 2.$$

With two rows there will be twenty cells connected in series, so that the resulting current is

$$C = \frac{nE}{R + \frac{n r_b}{m}} = \frac{20 \times 1{\cdot}4}{40 + \frac{20 \times 4}{2}}$$

$$= \frac{20 \times 1{\cdot}4}{80} = \frac{1{\cdot}4}{4} = 0{\cdot}35 \text{ ampere.}$$

Note. If the value found for m is not a whole number and exactly divisible into the total number of cells, the nearest possible value must be taken. Thus, if the internal resistance of each cell were 6 ohms, then

$$\frac{40}{m^2} \times 6 = 40$$

from which $$m = \sqrt{6} = 2{\cdot}44.$$

2 and 4 are factors of 40. If we take $m = 2$, C will be 0·28 ampere, whilst if $m = 4$ be taken, C becomes 0.254 ampere.

(3) Twelve Daniell's cells have to be arranged so that the maximum amount of heat may be produced in a conductor of 2 ohms resistance forming the external circuit. How must the cells be arranged to do this, and what will be the terminal P.D. of the battery and the number of watts employed in heating the conductor, if the E.M.F. and internal resistance of each cell be 1·07 volts and 1·5 ohms respectively?

To develop the maximum amount of heat in a wire of given resistance it is obvious, that the arrangement must be such as to give the maximum current through the external resistance, since the power available externally is given by C^2R. Consequently we must first determine the arrangement of the cells to give the maximum current. Let m = the number of rows connected in parallel, then $\frac{12}{m}$ = the number of cells in each row connected in series, and $\frac{1}{m} \times \frac{12}{m} \times 1{\cdot}5$ ohms is the internal resistance of the arrangement, and

$$\frac{12 \times 1{\cdot}5}{m^2} = 2 \quad \text{or} \quad m^2 = 9$$

and $$m = 3.$$

The cells must therefore be connected so as to form three rows connected

in parallel with four cells in each row connected in series. The resulting current is found as follows

$$C = \frac{4 \times 1{\cdot}07}{2 + \frac{4 \times 1{\cdot}5}{3}} = \frac{4 \times 1{\cdot}07}{4}$$

$$= 1{\cdot}07 \text{ amperes.}$$

The terminal P.D. is $e = CR = 1{\cdot}07 \times 2 = 2{\cdot}14$ volts, and the watts developed in the wire is $C^2R = (1{\cdot}07)^2 \times 2 = 2{\cdot}2898$ watts.

(4) Determine the least number of cells, and also how they must be arranged, so that 1 ampere may be sent through an external resistance of 15·6 ohms with an efficiency of 60 per cent. Given that the E.M.F. of each cell is 2 volts and its internal resistance 4 ohms.

Since the total E.M.F. is

$$E' = \frac{R}{\eta} \times C \quad \text{and} \quad \eta = {\cdot}6$$

$$E' = \frac{15{\cdot}6}{{\cdot}6} \times 1 = 26$$

Let n = number of cells connected in series

and m = number of rows of cells connected in parallel

then mn = total number of cells

but $$n = \frac{\text{total E.M.F.}}{\text{E.M.F. of each cell}} = \frac{26}{2}$$

$\therefore$ $n = 13$ cells in series.

Again, total internal resistance is

$$r_b' = \frac{R\,(1-\eta)}{\eta} = \frac{n \times r_b}{m}$$

$$\therefore \quad m = \frac{13 \times 4}{\frac{15{\cdot}6 \times (1 - {\cdot}6)}{{\cdot}6}} = \frac{52}{26 \times {\cdot}4}$$

$$= 5 \text{ rows.}$$

That is 13×5 or sixty-five cells, arranged in five rows with thirteen cells in a row are required.

(5) Five circuits are worked on the universal battery system, and each circuit has a resistance of 400 ohms. Determine the maximum resistance which the battery may have, so that in no case may the working current of any circuit vary more than 20 per cent.

Since the current in each branch is

$$c = \frac{E}{mr_b' + R}$$

we have

$$c_5 = \frac{E}{5r_b' + 400} \quad \text{(5 circuits in use)}$$

$$c_4 = \frac{E}{4r_b' + 400} \quad \text{(4 ,, ,,)}$$

$$c_3 = \frac{E}{3r_b' + 400} \quad \text{(3 ,, ,,)}$$

$$c_2 = \frac{E}{2r_b' + 400} \quad \text{(2 ,, ,,)}$$

$$c_1 = \frac{E}{r_b' + 400} \quad \text{(1 ,, ,,)}$$

To find r_b' so that c_5 and c_1 vary by not more than 20 per cent., we must solve the equation

$$\frac{\dfrac{E}{r_b'+400}}{\dfrac{E}{5r_b'+400}} = \frac{100}{80}$$

or

$$\frac{5r_b'+400}{r_b'+400} = \frac{5}{4}$$

$$\therefore \quad 4\,(r_b'+80) = r_b'+400$$

$$r_b' = \frac{80}{3} = 26\tfrac{2}{3} \text{ ohms.}$$

EXERCISES II E.

(1) Two Daniell's cells of E.M.F. 1·07 volts and internal resistance 1·5 ohms are available for sending a current through a conductor of 5 ohms resistance. Determine the resulting current when (1) only one cell is used, (2) when two cells connected in series are used, and (3) when two cells connected in parallel are used.

(2) How many cells, each 1·5 volts and 0·3 ohm resistance, must be connected in series to give 2·5 amperes through an external resistance of 7·5 ohms?

(3) Twelve cells, each 1·25 volts and 0·75 ohm, are connected in series. What will be the current flowing through an external resistance of 16 ohms if four of the cells are connected in opposition to the others?

(4) Determine the strength of the current when five cells are joined in parallel; the E.M.F. and internal resistance being respectively 1·4 volts and 3·5 ohms, and the external resistance ·55 ohm.

(5) A Daniell's cell, the internal resistance of which is 0·3 ohm, works through an external resistance of 1 ohm. What must be the resistance of another Daniell's cell, so that when it is joined up in series with the first and working through the same external resistance the current shall be the same as before? If the cells be joined up in parallel, how will the current be modified? (S. and A.)

(6) Twenty cells of E.M.F. 1·8 volts, connected in series, sends a current of 1·5 amperes through an external resistance of 12 ohms. How may they be otherwise connected so as to send the same current through the same resistance?

(7) Twenty cells, each 1·8 volts and 0·75 ohm resistance, are arranged in five rows of four in series. What is the resulting current if the external resistance is 1·2 ohms?

(8) Twenty-five cells of internal resistance 0·8 ohm, connected in

series, send 0·4 ampere through a resistance of 50 ohms. How many of these cells, also connected in series, would produce the same current through each of two wires of 75 ohms resistance connected in parallel?

(9) You are given sixteen cells, each having an internal resistance of 1 ohm. How would you connect them up so as to get as large a current as possible through a wire of 4 ohms resistance? If each cell had an electromotive-force of 1 volt, what current would you get through the wire? (C. and G.)

(10) A battery of twelve equal cells in series screwed up in a box, being suspected of having some of the cells wrongly connected, is put into circuit with a galvanometer and two cells similar to the others. Currents in the rates of 3 to 2 are obtained according as the introduced cells are arranged so as to work with or against the battery. What is the state of the battery? (S. and A.)

(11) Which would give the strongest current through a telegraph line of 30 ohms resistance—(*a*) twelve cells of a battery, the cells being all in series, or (*b*) eight cells arranged, four in series, and two rows in parallel? The resistance of each cell is 10 ohms. (C. and G.)

(12) One hundred cells, each having an E.M.F. of 1 volt, and an internal resistance of 5 ohms, have to be joined up so as to send the maximum current through an external resistance of 125 ohms. How would you connect them up, and what will be the current?

(13) A battery consisting of twenty-five cells, each 2 volts and 0·8 ohm, are connected in series, and the external resistance consists of five wires, each 150 ohms, connected in parallel. What is the efficiency of the arrangement?

(14) Determine the resulting current from a battery of forty cells each 1·8 volts E.M.F. and 4 ohms internal resistance through 40 ohms resistance, when (1) the cells are connected in series, and (2) when arranged with four rows in parallel and ten cells connected in series in each row. Also determine the efficiency of each arrangement.

(15) Six circuits work on the universal battery system, each circuit having a resistance of 500 ohms; what must be the greatest resistance which the battery can have, so that in no case the working current on any circuit may vary more than 25 per cent.? (C. and G.)

(16) How many wires of 1000 ohms resistance can be connected to a universal battery of 50 ohms resistance, so that the current given through each line shall not be less than 80 per cent. of the current which would flow if one line only were connected to the battery? (C. and G.)

(17) A battery is required to supply an activity of 160 watts to an external circuit of 1·6 ohms resistance; determine the minimum number of cells of E.M.F. 2 volts and internal resistance 0·1 ohm, which will yield this activity.

Section II. Heating Effects of a Current.

§ 26. **Heating Effects.** When electricity traverses a conductor from a point of higher potential to one of lower a *drop in volts* occurs equal to the difference of potential between the two points, and according to the definition of potential given in Section III, Chapter I, this difference of potential is measured by the number of units of work done on a unit of electricity in passing from the one point to the other. If a current C traverses the conductor, then C units of electricity pass per second, and if the drop in volts be E the work done per second is EC units. But in the system of practical units an amount of work, equal to the joule, is done when a coulomb of electricity is sent along a conductor from one point to another between the ends of which an electrical pressure of 1 volt is maintained, so that when a current of C amperes traverses a conductor of resistance, R ohms, under a pressure of E volts, the quantity of electricity which traverses the conductor every second is C coulombs, and EC joules of work are done per second. But when work is done at the rate of 1 joule per second, the rate of working, or the power, is equal to the watt, consequently the expenditure of power in the above case is EC watts.

If the portion of the circuit between the two points under consideration contains any source of counter-E.M.F., as when the current decomposes a chemical compound or drives an electric motor, a definite portion of the electrical pressure supplied will be utilized to neutralize the counter-E.M.F. set up, and less energy will be expended upon the conductor.

But when no energy is expended in any such way the EC units of work done by the electric forces during each second is spent in overcoming the resistance of the conductor, and according to the Principles of the Conservation of Energy, the amount of work cannot be lost but must give rise to an equivalent amount of some other kind of energy. Experiment indicates that this amount of electrical energy is converted into thermal energy, and it is found that whenever electricity traverses a conductor the temperature of the conductor is raised, indicating that the EC units of electrical energy is *frittered down into heat.*

Let H equal the number of units of heat which are equivalent to the conversion of CE joules per second into thermal energy in a con-

ductor of R ohms resistance. If these H units of heat were spent in doing mechanical work the amount of work done per second would be JH units, J being the *mechanical equivalent of heat*, i. e. the amount of work which 1 unit of heat is capable of doing when it is all converted into mechanical energy. In the C.G.S. or absolute system of units, the unit of heat is the *Calorie* and *is that amount* of heat which will raise the temperature of 1 gramme of water 1° C. The mechanical equivalent of the calorie has been carefully determined, and found to be approximately 42000000 ergs.

$$\therefore \qquad J = 4{\cdot}2 \times 10^7 \text{ ergs} = 4{\cdot}2 \text{ joules.}$$

The power expended in producing electrically H calories of heat in the conductor is

$$P = EC \text{ watts} = EC \text{ joules per second}$$

and the equivalent amount of thermal energy produced is

$$JH = 4{\cdot}2 \times 10^7 \times H \text{ ergs per second}$$
$$= 4{\cdot}2\, H \text{ joules per second}$$

$$\therefore \qquad 4{\cdot}2\, H = EC \quad \text{and} \quad H = \frac{EC}{4{\cdot}2} = 0{\cdot}24\, EC \text{ calories}$$

where H = number of calories. In t seconds the amount of work done will be

$$JH = ECt$$

and
$$H = \frac{EC\,t}{4{\cdot}2} = 0{\cdot}24\, EC\,t \text{ calories}$$

But by Ohm's Law $EC = C^2R = \dfrac{E^2}{R}$

$$\therefore \qquad H = 0{\cdot}24\, EC\,t = 0{\cdot}24\, C^2R\,t = 0{\cdot}24\frac{E^2\,t}{R} \qquad (53)$$

This theoretical relationship was first verified experimentally by Dr. Joule, and equation (53) is the algebraical expression of Joule's Law, which in words is :—

The number of units of heat developed in a conductor traversed by electricity is proportional

(1) *to the square of the current ;*
(2) *to the resistance of the conductor ;*
(3) *to the time that the current lasts.*

This production of heat and the elevation of temperature which results are absolutely definite effects of resistance which attend the passage of electricity through conductors, and the conversion of a portion of the electrical energy transmitted into thermal energy is the price paid for the transmission ; consequently the amount of energy available is less than that supplied. Some loss always attends the transformation and conversion of energy from one form to

another, and the heat developed electrically in a conductor is dissipated by conduction, radiation, and convection ; but it is very difficult to separate the amount of energy wasted by each of these causes, since so much depends upon the environment of the conductor, and the local circumstances generally. Of course, one would expect that as soon as the temperature of a wire exceeds that of the surrounding air and objects heat would be transferred from the conductor by one or more of the means just mentioned, but the temperature continues to rise until a certain limiting temperature is reached. As a matter of fact the temperature of the wire becomes stationary when the electrically generated heat balances the heat emitted by radiation, &c., or in other words when

the heat generated per second = heat lost per second.

In the case of bare overhead wires the loss by conduction may be neglected, and insulated and buried wires are kept cooler by the insulating covering, &c., since the radiation is increased by increasing the surface of the cable ; much, however, depends upon the temperature of the wire and the thermal resistivity of the covering, &c. Much experimental work has been done to determine the loss by radiation, but the laws are not simple nor properly known. We may, however, apply the principles of Newton's Law of Cooling and Dulong and Petit's Law, as being approximately applicable to the problem, and state that the loss by radiation is apparently proportional to

(1) the exposed surface of the conductor ;

(2) the excess of the temperature of the wire relatively to that of the surrounding medium ;

(3) the emissive power or emissivity, which is a quantity varying with the nature of the surface, degree of polish, &c., and upon the nature of the surrounding medium.

The emissive power is the amount of heat radiated per second per unit surface (1 square centimetre) per 1° C. of excess temperature, and is denoted by the symbol ϵ. The value of ϵ for dull or blackened surfaces is greater than for bright or polished surfaces.

The loss by convection is a more complex quantity than that by conduction or radiation, and as it depends upon factors which are functions of the surrounding medium and its movements, very little reliable information can be given respecting these losses, except that the convection loss is proportional to the exposed surface within certain limits.

To determine the law connecting the diameter d of a bare conductor with that strength of current C which raises its temperature by a fixed amount θ°C. above that of the surrounding air, and

maintains this excess of temperature stationary, we may proceed as follows.

Heat produced electrically per second $= H_t = \dfrac{C^2 R}{J}$

$$= \frac{C^2}{J} \cdot \frac{4 \sigma l}{\pi d^2} \quad . \quad . \quad . \quad (54)$$

where σ = resistivity of the wire at the stationary temperature.

Heat lost per second $= H_\theta = \pi d \epsilon \theta l$ . . . . (55)

When the temperature is stationary H_t and H_θ are in equilibrium

$$\therefore \quad H_t = H_\theta$$

and
$$\frac{C^2}{J} \cdot \frac{4 \sigma l}{\pi d^2} = \pi d \epsilon \theta l$$

$$\therefore \quad C^2 = \frac{\pi^2 d^3 J \epsilon \theta}{4 \sigma}$$

and
$$C = a d^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\ a)$$

$$d = \left(\frac{C}{a}\right)^{\frac{2}{3}} = a_1 C^{\frac{2}{3}}$$

where
$$a = \sqrt{\frac{\pi^2 J \epsilon \theta}{4 \sigma}}$$

also
$$\theta = b \frac{C^2}{d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56\ b)$$

where
$$b = \frac{4 \sigma}{\pi^2 J \epsilon}.$$

Also, current density $= \dfrac{\text{current}}{\text{area of cross-section}}$

and
$$\frac{C}{\text{area}} = \frac{C}{\frac{\pi}{4} d^2} = \frac{1}{\frac{\pi}{4} d^2} \sqrt{\frac{\pi^2 d^3 J \epsilon \theta}{4 \sigma}}$$

$$= \sqrt{\frac{4 J \epsilon \theta}{d \sigma}}$$

$$\therefore \quad \text{current density } \frac{C}{A} = \sqrt{\frac{\theta}{d}} \times \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where
$$\beta = \sqrt{\frac{4 J \epsilon}{\sigma}}.$$

The above are symbolic expressions of the following facts:—

(1) The elevation of temperature is independent of the length of the wire.

(2) The elevation of temperature is proportional to C^2 and also to $\dfrac{1}{d^3}$.

(3) A given conductor is only capable of carrying a certain current with a given elevation of temperature, and the current density is

proportional to the ratio of the square roots of the temperature excess and the diameter of the wire.

(4) For different sized wires of the same material raised to some stationary temperature, the squares of the currents ought to vary as the cube of the diameters of the wires, or

$$\frac{C_1}{C_2} = \left(\frac{d_1}{d_2}\right)^{\frac{3}{2}}.$$

This relation is not true if the wires are either extremely small or large; when the wires are small the convection loss is not proportional to the area of the exposed surface, whilst when the wires are large the temperature of the wire is not constant throughout the cross-section, being greatest at the centre. Professor Forbes states that experiment proves that the current varies more nearly as the diameter than as the diameter $\times \sqrt{\text{diameter}}$ for equal heating.

(5) If, as experiment indicates, the values of ϵ and σ vary with temperature in such a manner that their ratio remains sensibly constant, then for a given wire we have by (56 b)

$$\frac{\theta_1}{\theta_2} = \frac{b\,\dfrac{C_1^2}{d^3}}{b\,\dfrac{C_2^2}{d^3}} = \frac{C_1^2}{C_2^2}$$

and

$$\frac{C_1}{C_2} = \sqrt{\frac{\theta_1}{\theta_2}}.$$

In connexion with the permissible elevation of the temperature of conductors in practice, we may mention that the Institution of Electrical Engineers fixes 130° F. as the limiting temperature, consequently 35° C. or 63° F. may be taken as the permissible excess of temperature of a conductor in use, and although no absolute value of the emissivity constant can be given, we may assume ϵ to equal 0·00025; in which case equation (56 a) or $C = a\sqrt{d^3}$ reduces to the empirical formula

$$C = \sqrt{\frac{\pi^2 J \epsilon \theta}{4\sigma}}\sqrt{d^3} = \sqrt{\frac{(3{\cdot}1416)^2 \times 4{\cdot}2 \times 0{\cdot}00025 \times 35}{4 \times 0{\cdot}000001642\,(1 + {\cdot}0038\,\theta)}}\sqrt{d^3}$$
$$= 214\sqrt{d^3}$$

since $J = 4{\cdot}2$, $\sigma = 0{\cdot}000001642\,(1 + {\cdot}0038 \times 35)$. This, of course, is a theoretical value, and no account has been taken of casing. According to Kennelly, however, $C = 138\,d^{\frac{3}{2}}$ (when d is in centimetres) is a safe relationship to use for conductors in casing, &c., in practice.

The Institution of Electrical Engineers gives the following rules relating to the current to be carried by cables under different conditions of external temperature.

(1) The maximum current for situations where the external temperature is above 100° Fahr. may be calculated by means of the formula

$$\text{Log } C = 0{\cdot}775 \log A + 0{\cdot}301 \quad \text{or} \quad C = 2\,A^{0{\cdot}775}.$$

The maximum rise in temperature will be about 10° Fahr. on large sizes.

(2) The maximum current allowable in any situation may be calculated by means of the formula

$$\text{Log } C = 0{\cdot}82 \log A + 0{\cdot}415 \quad \text{or} \quad C = 2{\cdot}6\,A^{0{\cdot}82}.$$

The maximum rise in temperature will be about 20° Fahr. on large sizes.

In both cases C is given in amperes, and A in 1000ths of a square inch.

The actual temperature of a conductor carrying a current depends also upon the thermal capacity of the material forming the conductor, consequently the value of the specific heat of the material is an important factor in the determination of the temperatures of wires. *Specific Heat* is defined as the ratio of the amount of heat required to raise the temperature of a given mass of a substance 1° C. to that required to raise the same mass of water (at 4° C.) through 1° C.; in other words, the specific heat is the number of calories required to raise the temperature of the mass of 1 gramme of the substance 1° C. The specific heat for the more common metals used as conductors is given in the table on p. 354, from which it will be seen that the specific heat of copper is 0·0933, which means that 0·0933 calorie will raise 1 gramme of copper through 1° C., i. e. 9·33 per cent. of the heat required to raise 1 gramme of water 1° C.

The elevation of temperature of a conductor may therefore be said to depend upon

(1) the resistivity of the material;
(2) the specific heat of the material;
(3) the radiating surface and its nature;
(4) the emissivity of the conductor.

Now, since $H = 0{\cdot}24\,C^2 R t$

1 joule expended in heating 1 gramme of water increases the temperature ·24° C.

∴ 1 joule expended in heating 1 gramme of copper increases the temperature $\dfrac{\cdot 24}{\cdot 0933}$ ° C.

and the energy required to raise the temperature of 1 gramme of copper 1° C. is $\frac{\cdot 0933}{\cdot 24}$ or 0·38875 joule, assuming the metal retains all the heat supplied,

$\therefore$ energy required to raise w grammes of a substance of specific heat s through $\theta°$ C. $\left.\right\} = \frac{w s \theta}{\cdot 24} = 4{\cdot}2\, ws\theta$ joules.

This relation may be applied to problems connected with electric heating appliances and the cost of the energy supplied, as explained below. Then again, the considerations of injury to the insulation and possibility of fire limit the permissible elevation of temperature, and in practice advantage is taken of the rapid increase of the temperature, due to the heating effect of a large current, to make use of fuses or thermal cut-outs, as protecting devices for circuits and appliances against abnormal currents. Fuses consist of comparatively thin wires or strips of some metal or alloy of fairly high resistivity and low melting point, and are so designed that when the current exceeds their rated capacity sufficient heat is generated in the fuse over and above that dissipated by conduction, radiation and convection, so that the desired fusing effects result; in other words, the temperature of the fuse tends to exceed the melting temperature of the metal. The melting points of fusible metals and alloys are given in table A, page 354. Opinions differ as to the rating of the capacities of fuses, but many engineers adopt the rule that a fuse should be proportioned to blow with a 50 per cent. overload; thus, on a 20-ampere circuit the fuse should be designed to go when the current reaches 30 amperes. The relationship given by equation (56 a) enables the diameter of a fuse wire to be determined for a given current when the constant a in the equation $C = a\, d^{\frac{3}{2}}$ is known. Preece has given the following as the values of a for the commonly used fusible alloys.

TABLE OF FUSE WIRES.

Material.	Value of the constant a when the diameter is given in		
	Inches.	Centimetres.	Millimetres.
Copper	10244	2530	80
Aluminium	7585	1873	59·2
Platinum	5172	1277	40·4
German Silver	5230	1292	40·8
Platinoid	4750	1173	37·1
Iron	3148	777·4	24·6
Tin	1642	405·5	12·8
Lead	1379	340·6	10·8
Alloy of Lead and Tin 2 : 1	1318	325·5	10·3

Worked Examples. (1) Determine the amount of energy required to boil a pint of water in an electric kettle, the time required per kilowatt, and the cost at 2*d*. per unit; assuming the initial temperature to be 15°C.

Since energy required $= ws\theta \times 4{\cdot}2$ joules

and $s = 1, \quad \theta = 100° - 15° = 85°$ C.

$$w = (\tfrac{1}{8} \text{ gallons} \times 10) \text{ pounds}$$

$$= \frac{10}{8} \times 453{\cdot}6 \text{ grammes.}$$

Note. 1 gallon of water weighs 10 pounds and 1 lb. = 453·6 grm.

$\therefore$ Number of joules required $= \dfrac{4536}{8} \times 1 \times 85 \times 4{\cdot}2 = 202419$ joules.

As an example of the change of units another solution is given in which the British Thermal unit is used instead of the calorie. By definition

1 Br. Th. unit is the amount of heat required to raise 1 pound of water from 60° to 61° F. and 1° F. = $\frac{5}{9}$° C.

$\therefore$ 1 Br. Th. unit will raise 453·6 grammes of water 1° F.

but $453{\cdot}6 \times \dfrac{1}{{\cdot}24} \times \dfrac{5}{9}$ joules will raise 453·6 grammes of water 1° F.

$\therefore$ 1 Br. Th. unit is equivalent to $\dfrac{453{\cdot}6 \times 5}{{\cdot}24 \times 9}$ joules

and 1 pint of water weighs $= \dfrac{10}{8} = 1\tfrac{1}{4}$ pounds

$\therefore$ 1 pint of water requires $= 1\tfrac{1}{4}$ Br. Th. units per 1° F.

$$= \frac{5}{4} \times \frac{453{\cdot}6 \times 5}{{\cdot}24 \times 9} \text{ joules per } 1° \text{ F.}$$

$\therefore$ to raise 1 pint of water from 15° to 100° C. requires

$$= \frac{5 \times 453{\cdot}6 \times 5}{4 \times {\cdot}24 \times 9} \times 85 \times \frac{9}{5} = 200812 \text{ joules.}$$

To determine the time required to reach boiling point by the expenditure of 202419 joules, we have

joules = watts × time in seconds, therefore the time taken per kilowatt is

$$t = \frac{202419}{1000} = 202{\cdot}419 \text{ seconds}$$

$$= 3{\cdot}3736 \text{ minutes.}$$

This, of course, is the theoretical value since it has been assumed that there is no loss. In practice electric kettles have been constructed to use 1000 watts (say 10 amperes at 100 volts) for 3·7 minutes to boil 1 pint of water, consequently the efficiency is

$$\eta = \frac{3{\cdot}3736}{3{\cdot}7} = 91{\cdot}1 \text{ per cent.}$$

Again, 202419 joules = work done 1 kw. in 3·3736 minutes.

$\therefore$ 202419 joules $= \dfrac{1 \times 3{\cdot}3736}{60}$ B.T.U.

and the cost at 2*d*. per unit $= \dfrac{3{\cdot}3736 \times 2}{60} = 0.11245$ penny.

(2) A current of 100 amperes traverses 1000 feet of copper wire 75 mils in diameter. Determine (1) the temperature of the wire at the end of 1 second, and (2) at the end of 5 minutes, assuming no loss by radiation &c.

Given that the resistance of 1 mil-foot of wire is 9·8 ohms, and the original temperature is 15°C. Sp. gravity of copper = 8·78.

$$\text{Resistance is } R = \frac{9{\cdot}8 \times 1000}{(75)^2} = \frac{9{\cdot}8 \times 8}{45} \text{ ohms}$$

$$\begin{aligned} \text{the heat developed per sec.} &= {\cdot}24\, C^2 R \\ &= {\cdot}24 \times 100 \times 100 \times \frac{9{\cdot}8 \times 8}{45} \\ &= 4181{\cdot}3 \text{ calories} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \text{the heat acquired is } H_t &= \text{mass} \times \text{specific heat} \times \text{temp. rise} \\ \text{but} \quad \text{volume} &= (1000 \times 12 \times 2{\cdot}54)\,({\cdot}075 \times 2{\cdot}54)^2 \times {\cdot}7854 \\ &= 869 \text{ cubic cms. } (1 \text{ inch} = 2{\cdot}54 \text{ cms.}) \\ \therefore \quad \text{mass} &= 869 \times 8{\cdot}78 \\ &= 7630 \text{ grammes} \\ \text{and} \quad H_t &= 7630 \times {\cdot}0933 \times \theta \\ \therefore \quad \text{rise in temperature} &= \theta = \frac{4181{\cdot}3}{711{\cdot}879} \\ &= 5{\cdot}87^\circ \text{ C.} \\ \text{In 5 minutes } \theta_5 &= 5{\cdot}87 \times 300 \\ &= 1761^\circ \text{ C.} \\ \therefore \quad \text{actual temperatures (1)} &= 15^\circ + 5{\cdot}87^\circ = 20{\cdot}87^\circ \text{ C.} \\ (2) &= 15^\circ + 1761^\circ = 1766^\circ \text{ C.} \end{aligned}$$

Copper, however, melts at 1050° C.

(3) Determine the maximum temperature of the wire in the above example (2) if the omissive power be taken as 0·00025.

$$\begin{aligned} \text{Surface of wire} &= (1000 \times 12 \times 2{\cdot}54)\,({\cdot}075 \times 2{\cdot}54 \times \pi) \\ &= 18240 \text{ sq. cms.} \\ \text{Rate of radiation} &= 0{\cdot}00025 \times \text{surface} \times \theta \\ \text{Rate of generation} &= 4181{\cdot}3 \text{ calories} \end{aligned}$$

$$\therefore \quad \theta = \frac{4181{\cdot}3}{{\cdot}00025 \times 18240} = 917^\circ \text{ C.}$$

$$\begin{aligned} \text{Actual maximum temperature} &= 15^\circ + 917^\circ \\ &= 932^\circ \text{ C.} \end{aligned}$$

(4) A certain piece of metal, 0·5 mm. in diameter, is employed as a fuse. If the resistivity (per cm. cube) is 0·0000092 ohm, and the melting point of the metal be 1700°C., determine the current which will just fuse the wire. Also determine the value of the constant a in Preece's formula $C = a\, d^{\frac{3}{2}}$.

Heat developed per second per centimetre length is

$$H = C^2 R = C^2 \times \frac{{\cdot}0000092}{({\cdot}05)^2 \times {\cdot}7854} \times {\cdot}24$$

Rate of radiation per centimetre length is

$$H_\theta = 0{\cdot}00025 \times \text{surface} \times \theta$$

$$\therefore \quad C^2 \times \frac{{\cdot}0000092 \times {\cdot}24}{({\cdot}05)^2 \times {\cdot}7854} = 0{\cdot}00025 \times (1 \times {\cdot}05 \times 3{\cdot}1416)\,(1700^\circ - 15^\circ)$$

$$\text{whence} \qquad C = \sqrt{\frac{0{\cdot}00025 \times ({\cdot}05)^3 \times 3{\cdot}1416 \times {\cdot}7854 \times 1685}{{\cdot}0000092 \times {\cdot}24}}$$

$$= 7{\cdot}67 \text{ amperes}$$

Since $$7{\cdot}67 = a\sqrt{(0{\cdot}5)^3}$$

$$a = \frac{7{\cdot}67}{(0{\cdot}5)^{\frac{3}{2}}} \text{ for diameters in millimetres}$$

$$= 21{\cdot}7.$$

EXERCISES II F.

(1) A current of 150 amperes traverses a conductor of 4 ohms resistance; how many watts are expended in heating the conductor, and how many calories of heat are produced in one minute?

(2) A P.D. of 100 volts sends a current of 22 amperes through a conductor; at what rate is heat being developed in the conductor?

(3) If the heat produced per second in a wire of one ohm resistance by a current of one ampere is called a joule, what will be the resistance of a wire in which 63 joules are produced per second when the current is 3 amperes?

(4) The same current passes through two pieces of wire of the same material, the diameter of the first being four times that of the second. Compare the amounts of heat developed in equal lengths of the two wires.

(5) Determine the rise in temperature of a kilogramme of water due to the heat generated in a wire by a current of 5 amperes in 20 minutes, if the P.D. between the ends of the wire immersed is 7·5 volts.

(6) A current passes through a coil of wire immersed in a vessel containing 3 kilogrammes of water, and then through a copper voltameter. The resistance of the wire is 5 ohms, and it is found that the temperature of the water rises 10° C. per minute. How much copper is deposited per minute?

(7) Two wires, A and B, of the same gauge and material, 100 ft. and 150 ft. long respectively, are connected to the terminals of a battery. Compare the amounts of heat generated in the wires A and B.

(8) A divided circuit consists of two branches of copper wire of the same length connected in parallel, their cross-sections are as 1 : 3. Determine the relative amounts of heat produced in the two wires.

(9) An arc lamp takes 12 amperes at a P.D. of 50 volts, and burns for one hour. Determine the equivalent of the amount of energy absorbed in calories.

(10) A glow lamp has 60 ohms resistance, and is supplied with

current at a pressure of 60 volts. Determine the quantity of heat radiated per second by the filament if six per cent. of the energy is converted into light.

(11) A current of 100 amperes is passed through a coil of wire 10 ohms resistance immersed in 10 kilogrammes of water at an initial temperature of 20° C. How long will it take to raise the temperature of the water to 100° C.?

(12) A current of 10 amperes traverses a resistance of 5 ohms for 6 seconds, and another current of 6 amperes traverses a resistance of 7 ohms. During what time must the latter current flow in order that the amount of heat generated in the two cases may be the same?

(13) The poles of a battery of internal resistance B are connected successively by two wires whose resistances are R_1 and R_2 respectively; if $B = \sqrt{R_1 R_2}$, (*a*) show that the quantities of heat developed per second in each of the two wires are the same; (*b*) compare the amounts of heat produced in the battery in the two cases.

(14) Prove that when a current divides between two wires connected in parallel the total amount of heat generated in the two wires is less than it would be if the current were divided between them in any other proportion.

(15) A wire of resistance r connects A and B, two points in a circuit, the resistance of the remainder of which is R. If without any other change being made A and B are also connected by $(n-1)$ other wires, the resistance of each of which is r, show that the heat produced in the n wires will be greater or less than that produced originally in the first wire according as r is greater or less than $R\sqrt{n}$. (S. and A.)

(16) A wire 0·1 cm. in diameter carrying a current of 10 amperes is found to reach a steady temperature of 100° C. Assuming that the specific resistance of the material is $2{\cdot}1 \times 10^{-4}$ ohm per centimetre cube, and the value of J as $4{\cdot}2 \times 10^7$ ergs, determine the amount of heat emitted at 100° C. by a square centimetre of the surface. (S. and A.)

(17) If electric energy for heating purposes be supplied at 2*d.* per unit, find the cost of boiling a quart of water in five minutes, and the mean power required. Initial temperature of water 15° C. Efficiency of kettle 90 per cent. (C. and G.)

(18) Two wires of the same material but different diameters are raised by currents to the same temperature. Determine how the currents depend on the diameters. The diameters are 1·2 and 0·3 mm. respectively. A current of 8 amperes raises the first to a tem-

perature of 120° C. What current is required to do the same to the second wire?

(19) A current of 1·5 amperes passes through a column of mercury whose resistance is 0·5 ohm, and which weighs 20·5 grammes, and has a specific heat of 0·033. Determine the maximum temperature reached.

(20) A short piece of lead wire is used as a fusible cut-out, determine its diameter so that a current of 7·2 amperes may just fuse it, assuming the specific resistance of the lead to be 19·85 microhms, and the melting-point 335° C. The initial temperature is 15° C.

(21) Determine the value of the constant a in the formula $C = ad^{\frac{3}{2}}$ to correspond to the fuse in the last exercise.

(22) Determine the melting-point of a fuse $\frac{1}{10}$ inch in diameter and 1 inch long which fuses at 324 amperes, given that the value of the constant a is 10244.

(23) The current which fuses a copper fuse is twice that which fuses a platinum one; compare the diameters of the two fuses.

(24) A lead covered cable with a copper core of 5 mm. diameter carries a current; determine the least current which will melt the insulation covering in 15 minutes if its melting point is 75° C. The initial temperature is 15°C.

Section III. Chemical Effects of a Current.

§ **27.** *Chemical Effects.* In § 2, Chapter I, reference was made to the fact that an electric current can break up chemical compounds into their constituent elements, and some information was given respecting the chemical effects of a current. In this section the elementary theory of electrochemical action will be given more in detail, and it is worthy of remark here, that although a century has passed since the voltaic cell was discovered, and since Carlisle and Nicholson, in 1800, discovered that a current of electricity decomposes water slightly acidulated, it is only within comparatively recent years that the theory of electrochemical action has been made clear. All substances, as regards electrical work, may be divided into two classes, (1) those existing in the form of wires, plates, and liquids (including mercury and pure metals in the molten state) which are not decomposed by the passage of electricity through them, and (2) those chemical compounds, either in solution or fused, and termed

electrolytes, which are decomposed by the passage of electricity through them.

When conductors of the second class are interposed in an electric circuit and the current is led into, and out of, the electrolyte by conductors of the first class, which are then termed *electrodes*, electrolytic action results in virtue of the difference of potential which exists between the electrodes, and the constituents of the electrolyte, called by Faraday *ions* (*wanderers*), undergo separation and the two essentially distinct particles of the electrolyte move in opposite directions towards the two electrodes. These ions may separate in a free or isolated state, or they may separate from the electrolyte by combining with other bodies. The positive or leading-in electrode is termed the *anode*, whilst the negative electrode is called the *kathode*; the ions which travel towards the anode are called *anions*, whilst those which travel towards the kathode are termed *kations*. The process of electrolytic action or the chemical changes effected by a current of electricity is termed *electrolysis*.

Since the time of Faraday (1833) it has been known that equivalent quantities of all elements are always combined with the same quantity of electricity, and it was certainly a happy idea which led Faraday to determine quantitatively the chemical changes effected by the same quantity of electricity by sending the same current through a series of electrolytic cells containing different electrolytes, thus enabling him to formulate the fundamental laws of electrolysis known as Faraday's laws.

Faraday's Laws. I. If an electric current be passed through an electrolyte the amount of chemical decomposition which takes place in a given time is proportional to the quantity of electricity which passes through the electrolyte in that time, i. e. is proportional to the strength of the current.

II. If the same current be passed through different electrolytes in succession the amount of the electrolyte decomposed in each, and of the substances set free, are proportional to their chemical equivalents.

These principles have been expressed in slightly different forms, and it will be useful to include at this point the following statement by Helmholtz. 'The same quantity of electricity passing through an electrolyte either sets free or transfers to other combinations always the same number of valencies.' To this we may add Hittorf's remark that 'electrolytes are separated by electrolysis into the same atoms or atomic groups as they exchange in their chemical reactions with one another.'

The quantitative relationships defined by Faraday's laws are perfectly simple, and if the purely technical terms here defined are understood no difficulty should be experienced in tracing the connexions existing between the various factors. The *atomic weight* of an element is the weight of its atom compared with that of hydrogen which is taken as unity. The *valency* of an element is the displacing or combining power of its atom. The atoms are said to be mono-valent (dyads), di-valent (monads), tri-valent (triads), tetra-valent, penta-valent, or hexa-valent, according as they will combine with or displace one, two, three, four, five, or six atoms of hydrogen.

TABLE OF VALENCIES OF THE COMMON SALT RADICALS.

Monads.		Dyads.		Triads.		Tetrads.		Pentads.		Hexads.
Pos.	*Neg.*	*Pos.*	*Neg.*	*Pos.*	*Neg.*	*Pos.*	*Neg.*	*Pos.*	*Neg.*	*Pos.*
H	Cl	Ba	O	As	PO_4	Sn(ic)	SiO_4	As	N	Al_2
K	Br	Ca	S	Sb	AsO_3	Pt		Sb		Fe(ic)
Na	I	Mg	SO_4	Bi	AsO_4	Al		Bi		Mn_2(ic)
	F	Zn	MnO_4	Au	BO_3	Pb		P		
NH_4	Cy	Cd		Fe(ic)	N					
Ag	HO	Hg(ic)								
Hg(ous)	NO_2	Pb								
		Cu(ic)								
		Fe(ous)								
		Mn(ous)								
		Sn(ous)								

The *chemical equivalent* of an element is its weight relative to hydrogen which enters into combination, and numerically the chemical equivalent of an element is obtained by dividing the atomic weight of the element by its valency (see table, page 122)

$$\text{or chemical equivalent} = \frac{\text{atomic weight}}{\text{valency}}.$$

The laws of electrolytic action may be expressed symbolically as follows:

Let ϵ_0 = number of grammes of one metal set free in unit time by unit current.

and ϵ = number of grammes of another metal set free in unit time by unit current.

m_0 and m = the atomic weights of the two metals respectively.

v_0 and v = the valencies.

Then
$$\frac{\epsilon}{m/v} = \frac{\epsilon_0}{m_0/v_0}$$
and
$$\epsilon = \frac{m}{v} \times \frac{v_0}{m_0} \times \epsilon_0.$$

If ϵ_0 be the number of grammes of the substance taken as the standard of reference whose atomic weight and valency are unity (i.e. if the substance be hydrogen) which are set free by unit current in unit time, then the number of grammes of the other substance set free by unit current in unit time is

$$\epsilon = \frac{m}{v}\epsilon_0 = \frac{\epsilon_0}{v/m}.$$

If the current be C units in strength and lasts for t seconds, then the weight of the metal deposited is

$$W = C\epsilon t = Ct\frac{m}{v}\epsilon_0 = Q\frac{m}{v}\epsilon_0$$

but by definition $\frac{m}{v}$ is the chemical-equivalent of the element, and ϵ the weight in grammes of an element set free by unit quantity of electricity is termed the electrochemical equivalent (E.C.E.) of the element, and it is obvious from the relation, $\epsilon = \frac{m}{v}\epsilon_0$, that the electrochemical equivalent of any element may be obtained by taking the product of the electrochemical equivalent of hydrogen by the chemical equivalent of the element (i.e. $\frac{m}{v}$*), or

$$\text{E.C.E. of an element} = \text{E.C.E. of hydrogen} \times \frac{\text{atomic weight of element}}{\text{its valency}}$$

thus
$$\text{the E.C.E. of oxygen} = 0{\cdot}000010384 \times \frac{15{\cdot}96}{2}$$
$$= 0{\cdot}00008296 \text{ grammes.}$$

The E.C.E. of hydrogen has been frequently determined by experiment, and in a general sense it may be looked upon as the fundamental quantity in electrochemical calculations, since the E.C.E. of any element may be readily calculated if its chemical-equivalent be known. The following table gives the atomic weights, valencies, &c., of the most common elements.

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* The chemical equivalents are ratio-numbers, whilst the electrochemical equivalents are numbers denoting the weights of the elements set free.

TABLE OF ELECTROCHEMICAL EQUIVALENTS, &c.

Elements and Symbols.		Atomic Weight.	Valency.	Chemical-Equivalent.	E.C.E. Grammes per Coulomb.	Coulombs per Gramme.
Electro-positive.						
Hydrogen . . .	H	1	1	1	0·000010384	96540·0
Potassium . . .	K	39·03	1	39·03	0·0004053	2469·0
Sodium	Na	23·	1	23·	0·0002388	4189·0
Gold	Au	196·2	3	$\frac{Au}{3} = 65{\cdot}4$	0·0006791	1473·0
Silver	Ag	107·67	1	107·67	0·0011181	894·5
Copper(ic) . . .	Cu	63·18	2	$\frac{Cu}{2} = 31{\cdot}59$	0·0003281	3050·0
,, (ous)		63·18	1	63·18	0·0006562	1525·0
Mercury(ic) . . .	Hg	199·8	2	$\frac{Hg}{2} = 99{\cdot}9$	0·0010374	964·3
,, (ous)		199·8	1	199·8	0·0020748	482·2
Tin(ic)	Sn	117·8	4	$\frac{Sn}{4} = 29{\cdot}45$	0·0003058	3283·
,, (ous)		117·8	2	$\frac{Sn}{2} = 58{\cdot}9$	0·0006116	1641·6
Iron(ous). . . .	Fe	55·9	2	$\frac{Fe}{2} = 27{\cdot}95$	0·0002902	3448·
,, (ic)		55·9	3	$\frac{Fe}{3} = 18{\cdot}63$	0·0001935	5171·
Nickel	Ni	58·6	2	$\frac{Ni}{2} = 29{\cdot}3$	0·0003043	3287·0
Zinc	Zn	64·9	2	$\frac{Zn}{2} = 32{\cdot}45$	0·00033698	2970·0
Lead	Pb	206·4	2	$\frac{Pb}{2} = 103{\cdot}2$	0·0010716	933·7
Electro-negative.						
Oxygen	O	15·96	2	$\frac{O}{2} = 7{\cdot}98$	0·00008296	12070·0
Chlorine	Cl	35·37	1	35·37	0·0003673	2724·0
Iodine	I	126·54	1	126·54	0·0013140	761·4
Bromine	Br	79·76	1	79·76	0·0008282	1208·0
Nitrogen	N	14·01	3	$\frac{N}{3} = 4{\cdot}67$	0·00004849	20630·0

§ 28. **Thermo-chemical Relations.** It is of course obvious that whenever electrolytic actions occur it is a very important matter to determine the minimum electrical pressure corresponding to the decomposition set up. This and the allied problem of determining the E.M.F. produced with a certain combination of elements, as in the case of voltaic cells, is purely a thermo-chemical question. Thus whenever a chemical reaction or definite chemical combination occurs energy is either absorbed or liberated in the form of heat, but in the

voltaic circuit this heat is mostly transformed and not evolved. When no mechanical work is done

heat producing work = work corresponding to the heat produced in the circuit + heat absorbing chemical work

or symbolically

$$H = \frac{CE}{J} + h. \qquad . \qquad . \qquad . \qquad (A)$$

In the heat-producing chemical work some substance must be changing its state of combination. Let us suppose that when one gramme of this substance undergoes the change in question h_1 units of heat are produced, and in like manner, at the point where chemical work is done, let us suppose that when one gramme of a different substance which is there undergoing change absorbs h_2 units of heat, then the above equation may be written as follows:

$$h_1 (\epsilon_0 C \frac{M}{V}) = \frac{CE}{J} + h_2 (\epsilon_0 C \frac{m}{v}) \qquad . \qquad . \qquad (B)$$

and

$$\epsilon_0 \frac{M}{V} h_1 = \frac{E}{J} + \epsilon_0 \frac{m}{v} h_2. \qquad . \qquad . \qquad . \qquad (C)$$

Now $\frac{M}{V} h_1$ and $\frac{m}{v} h_2$ are quantities which may be defined as the intensities of the electrical energy evolved or absorbed by the electro-chemical formation or separation of a compound, and values of the heat of formation for a number of salts of the more common metals are expressed in calories in the accompanying table.

Let $\frac{M}{V} h_1 = Q$ and $\frac{m}{v} h_2 = q$, then

$$\epsilon_0 Q = \frac{E}{J} + \epsilon_0 q$$

and

$$E = J \epsilon_0 (Q - q).$$

In practical units ϵ_0 is the E.C.E. of hydrogen and $= 0{\cdot}000010384$, and $J = 4{\cdot}2$ joules.

$$\therefore \quad J\epsilon_0 = 0{\cdot}0000436$$

$$\therefore \quad E = 0{\cdot}0000436 (Q - q). \qquad . \qquad . \qquad (D)$$

Therefore if in a circuit two chemical actions are going on simultaneously, one of which tends to produce, and the other to absorb heat, the E.M.F. in volts set up may be calculated by subtracting the amount of heat produced per chemical equivalent in heat-absorbing action from that produced in the heat-producing action, and multiplying the difference by 0·0000436.

The heats of formation given in the following table represent the chemical energy of the quantity of each substance, which is electro-chemically equivalent to one gramme of hydrogen, that is, the energy

evolved or absorbed in an electrochemical reaction during the passage of an electric current, which simultaneously liberates one gramme of hydrogen in a voltameter. The constant 0·0000436 may also be obtained by dividing 4·2 (the joules equivalent to a gramme-calorie) by 96540, the number of coulombs required to liberate one univalent gramme-electrochemical equivalent of any substance *. It will be noticed that the number of coulombs required to liberate one gramme of the various elements is given in the sixth column of the table on page 122.

TABLE OF HEATS OF FORMATION AND SEPARATION.

Compounds.	Ions.	Calories.	Compounds.	Ions.	Calories.
KCl	K and Cl	101200	CuO	Cu and O	18850
$NaCl$	Na ,, Cl	96600	HgO	Hg ,, O	10750
$ZnCl_2$	Zn ,, Cl_2	56500	Na_2SO_4	Na_2 ,, SO_4	93700
HCl	H ,, Cl	39400	$ZnSO_4$	Zn ,, SO_4	53450
$AgCl$	Ag ,, Cl	29000	$FeSO_4$	Fe ,, SO_4	46900
$CuCl_2$	Cu ,, Cl_2	31250	H_2SO_4	H_2 ,, SO_4	34500
$HgCl_2$	Hg ,, Cl_2	25150	$PbSO_4$	Pb ,, SO_4	37300
ZnO	Zn ,, O	41750	$CuSO_4$	Cu ,, SO_4	28200
H_2O	H_2 ,, O	34500	$HgSO_4$	Hg ,, SO_4	1200
PbO	Pb ,, O	25400			

It is interesting to add that Grotthus in 1805 put forward the theory that the atoms of each molecule of a compound were charged with different kinds of electricity, some with positive and the others with negative, and that when a certain difference of potential existed the positive atoms all moved to the anode, and the negative ones formed a stream of atoms moving to the kathode. The modern views respecting the phenomena of electrolysis, and which give rise to what is known as the electronic theory, are based upon the hypothesis that electricity may be regarded as having atomic structure, and that the atom of electricity, or *electron* as it is called, is an electric-charge-carrying corpuscle much smaller than the chemical atom. As Helmholtz remarks, 'every single valency of an element or compound ion is charged with exactly the same quantity of positive or negative electricity which behaves as if it were an electrical atom that cannot be further divided.' Some electrons carry positive charges and others negative, and it is probable that the electrons of the chemical elements are the same size and possibly identical. The ions may therefore be regarded as the chemical

* 96540 coulombs is the quantity of electricity which deposits that number of grammes which is equal to the 'chemical equivalent' of the element.

combinations of the element with such an electron, or an ion is a saturated chemical combination of an atom and an electron. A sodium ion, for instance, is entirely different from a sodium atom, since it is a combination of a sodium atom with a positive electron. All elements have affinity to positive as well as negative electrons with which they form ions.

A hydrogen atom may thus be decomposed into a very large number of electrified corpuscles or electrons, and the action of dissolving a salt in water results in the partial dissociation and formation of free ions with electric charges. Chemical affinity may be explained by this theory by considering that the electrons are the chemical bonds which tie together, as it were, the atoms into the compound molecule, and when positive and negative elements combine there is, besides the combination of the two atoms, also a combination of a negative and positive electron.

Worked Examples. (1) Determine the strength of current required to decompose 1 gramme of water in 5 minutes, and state the volumes of hydrogen and oxygen produced.

Since $$W = C\epsilon t$$
the weight of hydrogen disengaged in the 5 minutes is
$$W_H = C \times 0{\cdot}000010384 \times 5 \times 60$$
but the weight of water decomposed in the same time is 9 W_H

but $9\,W_H = 1$ gramme $\quad\therefore\quad \frac{1}{9}$ gramme $= {\cdot}0031152\ C$

and $$C = \frac{1}{{\cdot}0280368} = 35{\cdot}6 \text{ amperes.}$$

Neglecting temperature effects, we may take the weight of 1 c.c. of hydrogen to be 0·00008988 gramme, $\therefore$ volume of $\frac{1}{9}$ grm. of H $= \frac{1}{9} \div 0{\cdot}00008988 = 1236{\cdot}2$ c.cms. and volume of O $= \frac{1}{2}$ vol. of H $= 618{\cdot}1$ c.cms.

(2) How long will it take 100 amperes to produce 2·5 grammes of caustic soda from a solution of Na Cl?

Since caustic soda = Na H O and the atomic weights of Na, H and O are 23, 1 and 16 respectively,

every gramme of caustic soda contains $\dfrac{23}{23+1+16}$ gramme of sodium.

$\therefore$ sodium required for 2·5 grammes of caustic soda
$$= \frac{23}{40} \times 2{\cdot}5 = 1{\cdot}4375 \text{ grammes}$$

and since $$W = C\epsilon t$$
$$1{\cdot}4375 = 100 \times 0{\cdot}0002388 \times t$$
$$\therefore \quad t = \frac{1{\cdot}4375}{0{\cdot}02388}$$
$$= 60{\cdot}2 \text{ seconds or 1 minute (approx.).}$$

(3) Determine approximately the E.M.F. of a Daniell's cell.

The chemical reactions taking place are

$$(1)\ Zn + H_2SO_4 = Zn\,SO_4 + H_2$$
$$(2)\ Cu\,SO_4 + H_2 = H_2SO_4 + Cu.$$

Thermochemical relations

$$(1)\left\{\begin{array}{lr} \text{Formation of } Zn\,SO_4 \text{ liberates} & 53450 \text{ calories} \\ \text{Separation of } H_2SO_4 \text{ absorbs} & 34500 \text{ calories} \end{array}\right.$$
$$\text{Heat available} \quad 18950 \text{ calories.}$$

$$(2)\left\{\begin{array}{lr} \text{Formation of } H_2SO_4 \text{ liberates} & 34500 \text{ calories} \\ \text{Separation of } Cu\,SO_4 \text{ absorbs} & 28200 \text{ calories} \end{array}\right.$$
$$\text{Heat available} \quad 6300 \text{ calories.}$$

Total amount of heat available for transformation is $18950 + 6300 = 25250$ calories.

$$\begin{aligned} \text{But } E \text{ (volts)} &= 0{\cdot}0000436 \times \text{heat available} \\ &= 0{\cdot}0000436 \times 25250 \\ &= 1{\cdot}1009 \text{ volts} \end{aligned}$$

Note. By application of formula (D) the same result is obtained more readily, thus

$$\begin{aligned} E &= 0{\cdot}0000436\,(Q - q) \\ &= 0{\cdot}0000436\,(53450 - 28200) \\ &= 1{\cdot}1009 \text{ volts.} \end{aligned}$$

(4) Determine the weight of zinc used in a Daniell's cell in 1 hour if the total resistance of the circuit is 7 ohms.

$$\text{E.M.F. of Daniell cell} = 1{\cdot}1009 \text{ volts (see No. 3)}$$

$$\therefore \text{ current} \quad = \frac{1{\cdot}1009}{7} \text{ ampere.}$$

$\therefore$ weight of zinc entering in combination in 1 hour is

$$\begin{aligned} W &= \frac{1{\cdot}1009}{7} \times 0{\cdot}00033698 \times 3600 \\ &= 0{\cdot}19079 \text{ gramme.} \end{aligned}$$

EXERCISES II G.

(1) If 0·0104 milligramme of hydrogen is evolved by one coulomb of electricity, determine the electrochemical equivalent of lead.

(2) A current of 5 amperes is passed through an acidulated water voltameter, determine the weight of hydrogen and oxygen evolved per hour.

(3) How much silver will 10 amperes deposit per minute?

(4) A battery deposits 0·1145 gramme of copper in 9 hours, determine the strength of the current.

(5) How long will it take 10 amperes to deposit 2·5 grammes of copper?

(6) A copper bath and a water voltameter are connected in series, and a certain current sent through the circuit. When 0·14365 gramme of copper is deposited it is found that 51 cubic centimetres of hydrogen have been evolved. If one cubic centimetre of hydrogen weighs 0·00008988 of a gramme, determine the electrochemical equivalent and chemical equivalent of copper.

(7) Determine the quantity of acidulated water which will be decomposed by a current of 12 amperes in 10 minutes.

(8) If 10000 kilocoulombs are passed through cells arranged in series containing solutions of $Cu_2\,Cl_2$; $Hg_2\,(NO_3)_2$; $CuSO_4$; H_2SO_4 and NaCl, how much copper, mercury, hydrogen, and caustic soda can be obtained? (C. and G.)

(9) What is Faraday's law regarding electro-deposition? How much caustic soda is produced per ampere-hour, and how much lead, silver, and mercury (from mercurous nitrate) would be deposited per ampere-hour? One coulomb evolves say 0·0104 milligramme of hydrogen, and the atomic weights of sodium, lead, silver, and mercury are respectively 23, 207, 108 and 200. (C. and G.)

(10) 1·25 amperes are used for silverplating for 15 minutes, determine the thickness of the deposit if the silver is deposited on a plate with 250 square centimetres of surface. Sp. gravity of silver 10·5 grammes per cubic centimetre.

(11) A water voltameter and a silver bath are connected in series and a current is sent through them for a certain time; determine the volumes of hydrogen and oxygen set free in the water voltameter if 100 grammes of silver are deposited.

(12) Determine the amount of copper used in 1 hour in a Daniell's cell if the total resistance of the circuit is 12 ohms.

(13) Determine the volume of mixed gas (hydrogen and oxygen) produced per second per ampere by the electro-decomposition of water.

CHAPTER III

PRINCIPLES OF DISTRIBUTION AND THE DESIGN OF MAINS

Section I. General Principles.

§ 29. **Introduction.** The distribution of electricity comprehends the study of the best means of disposing electrical energy economically to a certain number of receivers. This study includes the consideration of the choice of a system of distribution from a variety of systems, to suit the particular requirements and needs of the case; the expenditure of capital and loss of energy; the control, regulation, and permanence of the working conditions of the conductors forming the network. It is thus evident that the problem of distribution is one of the most important problems encountered by the electrical engineer.

The systems of distribution vary principally by reason of the different methods of arranging the conductors adopted to render efficient service. The simplest arrangement is the *series* system, which lends itself naturally to arc lighting, whilst with the altered conditions, which were introduced when incandescent lighting was adopted, the series system gave way to the simple *parallel* or *two-wire* system. With the former system we have a *constant current circuit*, whilst with the latter we have a *constant potential circuit.* And it is in consequence of the simple two-wire system not satisfying all the conditions for efficient working, when the distribution is on a large scale, as for town lighting for instance, that the three-wire system and various modifications have been introduced.

§ 30. **Fundamental Principles.** In all cases in practice where an extended area has to be lighted, the difficulty encountered by the engineer is that of increasing the range of distribution, i. e. the permissible distance between the generator and the lamps, without undue loss or variation of pressure, or violating the laws of economy of conductors. The general method is to decide upon a certain number of *centres of distribution* at comparatively short distances apart, as for instance at the junctions of main streets; and by connecting these nodes or centres together so as to enclose a certain area we have a representation of a net or mesh, more or less regular. The whole of such a system of mains, together with the service mains

(supplying the consumers) connected with them, forms a distributing network. To feed such a network the centres of distribution are connected to the generators by a series of special feeding mains which divide the current supplied, and are of sufficient section to carry the current required at the distributing centre. *These feeding mains are termed feeders,* and by French writers *artères* (arteries), which is a very significant and suggestive term. A simple network supplied by feeders is shown in Fig. 19, in which F, the feeders, run from the generating station to C the centres of distribution, D are the distributing mains or distributors, and S M the service mains.

The function of a feeder is that of a carrier, and it is evident that they may be long or short to suit the circumstances, and as they are only used for transmission of electrical energy, they are never tapped by distributors. It is also obvious that if the feeders for a network have been properly designed so that the drop in pressure in them shall be the same, then the centres of distribution will all be at the same potential. These points may thus be looked upon as secondary sources or centres of feeding maintained at a constant potential, distributing the current into the network. The pressure at these centres is maintained constant by regulating appliances at the generators. The section of the feeders is determined by considering the maximum current which traverses them and their length, so that the permissible drop in pressure is not exceeded.

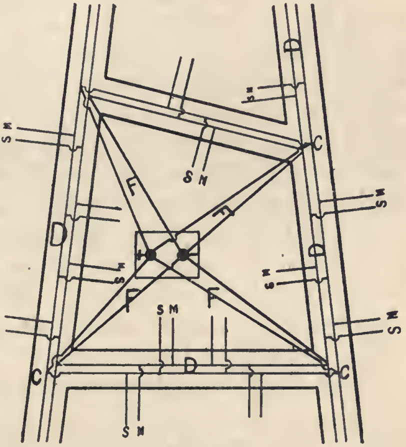


Fig. 19.

The distributors distribute the current received from the feeders to the service mains, and are therefore tapped at all points where current is required. They vary in size and length according to circumstances, and in some cases do not form a network as shown in Fig. 19, but a series of radiating mains. Unlike feeders, the current carried by the distributors is not the same at all places.

§ 31. **Question of Voltage.** When we come to consider the transmission and distribution of electrical energy quantitatively and from the point of view of economy, we find the problem complicated and difficult of solution. Mr. Blackie, in his paper read before the Municipal Electrical Association in 1897, put the whole thing in

a nutshell when he said, 'The most economical and efficient means of transferring electrical energy from the generating source to indefinite points, in indefinite quantities and at variable times, is beyond human calculation.' Many of the conditions which have to be satisfied before the most economical and efficient system can be devised are mutually so contradictory and conflicting that no hard and fast rule can be given even for the solution of the general problem. But experience has taught many lessons, and the fundamental principles of the various systems are now pretty well known, and the notes in this chapter are intended to give an insight into the principles which have to be considered when designing the transmission and distribution part of a system of electrical supply.

The number of systems which have been proposed and tried is great, but as in other cases of rapid development, a process of evolution has been at work, and we may consider that there are two main systems, each of which may be sub-divided into a number of others, which are more or less modifications of one another. We have, for instance, (1) the most primitive and simple method of connecting the receiving mechanism directly with the generator, and (2) the method in which there is no direct electrical connexion between the source and the receiver. The first is, of necessity, a low-pressure system, and is usually a continuous current system of supply; the latter is a high-pressure system, with single or double conversion or transformation, and is, almost of necessity, an alternating current system employing transformers.

In the first method, conversion or transformation of pressure is absent, and the E.M.F. of the generator is limited by the voltage of the receiving mechanism, plus a small percentage of this voltage to compensate for the 'line drop in volts'; for lighting circuits, other than series arc lighting circuits, the maximum pressure is fixed at about 500 volts, whilst for motor circuits 500 to 600 volts will be the limit for continuous currents. With transformation, however, the pressures adopted are much higher and irrespective of the voltage required, the usual magnitude being of the 10,000 to 20,000 volts order, with double conversion, and from 2,000 to 3,500 volts with simple reduction transformation. It must be remarked, however, that for distribution to the consumer there is not much difference in the magnitude of the pressure supplied, and that the second method includes transmission as well as distribution. Another point of importance to be noticed is that with direct current machines the limit of pressure which may be generated is soon reached, in consequence of commutation and the difficulties attending the insulation of the armature conductors, whilst with alternators the

limit of the pressure is fixed by the working of the line, and is perfectly independent of the construction and working of the machine. The present tendency, abroad, at least, is to increase the pressure for transmission purposes, and in one case the pressure in use is 60,000 volts.

Neither method is the ideal one for all cases, nor is one more efficient and economical than the other in all cases. Before an engineer can decide which system to adopt, therefore, he must take into consideration the characteristics of each system and the nature of the demand. The selection, in fact, depends upon (1) the area and the character of the district to be supplied from one generating station, (2) the position of the generating station relatively to the area of supply, and (3) the nature and cost of the power available for driving the generators. Presently we shall deduce a number of general statements from first principles, which will prove that the above factors decide in the main the choice of the system. Other factors, of course, have to be considered, the most important of which are—capital outlay on the mains and cost of maintenance, flexibility and capability of subsequent extension and development, regulation, uniformity, and reliability of pressure, and the probable dissipation of energy, leakage, and losses due to resistance, hysteresis, &c.

In many respects the first of these factors, i. e. the cost of mains, is the most important. When considered as an investment it is obvious that the interest thereon forms a very considerable portion of the fixed annual charges against the system. The first cost increases—other things being equal—with the distance over which the power, P, has to be transmitted, but not directly, as we shall show, and it also depends largely upon the voltage employed.

The line losses are given by C^2R, consequently *for a given length and weight (or cross-sectional area) of the cable the line loss varies as the square of the current.*

Let $C^2R = w$ watts, then

$$\text{the efficiency of transmission} = \frac{P}{P+w} = \eta$$

$$= \frac{E_1}{E_1 + e},$$

where e is the drop in volts in the line, and E_1 the voltage at the receiver end.

$$\text{But} \qquad R = \sigma\frac{2L}{A} \text{ and } R = \frac{e}{C},$$

$$\therefore \quad A = \frac{\sigma\, 2L \,.\, C}{e}$$

in which L = distance between the generator and the centre of distribution, therefore, for *given values of E_1, e, and P, i. e. at a fixed efficiency, the weight and cost of the conductor varies proportionately to the square of the distance.* Thus, if the distance L in one case be increased to nL in another case, we have for the two cases and given values of E_1, e, and P (i. e. fixed efficiency) the cross-sectional areas

$$A_1 = \sigma \frac{2L \cdot C}{e} \quad \text{and} \quad A_2 = \frac{\sigma\, 2nL \cdot C}{e},$$

$$\text{and } A_2 : A_1 = nL : L,$$

or, in words, the area of the conductor must increase directly as the distance to satisfy the conditions given, and the weights of copper are

$$W_1 : W_2 = 2L \cdot A_1 : 2n\,L \cdot A_2$$
$$= L^2 : (nL)^2,$$

or, in words, the weight and cost of the conductor vary directly as the square of the distance.

To determine how the voltage employed affects the choice of the system, let us consider the transmission of a given amount of power, P, both by a low-pressure system and a high-pressure system. The various factors introduced will be denoted by a symbol with one dash in the case of the low-pressure system, and by a symbol with two dashes for those of the high-pressure system. And let $E_2 = nE_1$,

then $$P = E_1 C_1 = E_2 C_2$$

or $$\frac{C_1}{C_2} = \frac{E_2}{E_1} = \frac{nE_1}{E_1} = n$$

and $$C_1 = nC_2.$$

If the loss in the mains and the distance remain the same, we have

$$C_1^2 R_1 = C_2^2 R_2\,;$$

and substituting values of C_1, R_1, and R_2

$$(nC_2)^2 \sigma \frac{2L}{A_1} = C_2^2 \sigma \frac{2L}{A_2},$$

or $$n^2 A_2 = A_1\,;$$

and the weights of copper are

$$W_1 : W_2 = 2L\,A_1 : 2L\,A_2$$
$$= n^2 : 1 = 1 : \frac{1}{n^2} \text{ or } E_2^2 : E_1^2\,;$$

therefore, *for the same loss in the mains over the same distance, the weight of copper required (and ∴ the cost of copper) varies inversely as the square of the voltage.*

There is another case to be considered. If the voltage be in-

creased in direct proportion to the distance, and the efficiency remain the same, then

$$L_2 = nL_1 \quad E_2 = nE_1 \quad \text{and} \quad e_2 = ne_1$$

and

$$C_2R_2 = n(C_1 R_1)$$

$$C_2 \sigma \frac{2 \,.\, nL_1}{A_2} = n \,.\, nC_2 \,.\, \sigma \frac{2L_1}{A_1}$$

or

$$A_1 = nA_2$$

and

$$\begin{aligned} W_1 : W_2 &= 2L_1A_1 : 2L_2A_2 \\ &= 2L_1 \,.\, nA_2 : 2 \,.\, nL_1 \,.\, A_2 \\ &= 1 : 1; \end{aligned}$$

therefore, as a general rule, *by increasing the voltage to be employed in direct proportion to the distance, the weight of copper required* (*and* $\therefore$ *the cost of copper*) *will be a constant quantity independent of the distance for a fixed efficiency.*

The following table gives the general relations existing between the various factors of a transmission scheme in which a given power P has to be transmitted a given distance when the pressures at the distributing points are E and nE, and the areas of cross-section of the conductors are A and $\frac{1}{m}$ A.

	At lower pressure E.	At higher pressure nE.
Current	$C_1 = \frac{P}{E}$	$C_2 = \frac{P}{nE} = \frac{1}{n}C_1$
Resistance of line	R_1	$R_2 = mR_1$
Cost of line	KA	$\frac{K}{m}A$
Drop in volts	$e_1 = C_1R_1$	$e_2 = C_2R_2 = \frac{1}{n}C_1 \times mR_1$ $= \frac{m}{n}e_1$
Power wasted	$w_1 = e_1C_1$	$w_2 = e_2C_2 = \frac{m}{n}e_1 \times \frac{C_1}{n}$ $= \frac{m}{n^2}w_1$
Percentage drop	$p_1 = 100\frac{e_1}{E}$	$p_2 = 100\frac{e_2}{nE} = 100\frac{m}{n^2}\frac{e_1}{E}$ $= \frac{m}{n^2}p_1$
Efficiency of transmission	$\eta_1 = \frac{e_1}{E}$	$\eta_2 = \frac{e_2}{nE} = \frac{m}{n^2}\frac{e_1}{E}$ $= \frac{m}{n^2}\eta_1$

Therefore, to secure the same efficiency in both cases, η_1 must be equal to $\frac{m}{n^2}\eta_1$, or $n^2 = m$ and $n\mathrm{E} = \sqrt{m}\,\mathrm{E}$; and if the distance between the generator and the centre of distribution exceeds a certain limit, the working voltage is a factor of paramount importance as regards the cost of the line. In the final solution, however, the cost of the transforming devices must be included in the prime cost, and the transformer losses cannot be neglected. There is, however, no doubt that the economical distribution and long distance transmission of electrical energy can be accomplished by the employment of high voltages, and has been rendered possible by the development of alternating and polyphase current systems and appliances. Transmission and distribution of alternating and polyphase currents will be dealt with in the second volume.

§ **32. The General Problem.** The difficulty of solving the problems in the design of mains is rendered the greater because some of the conditions imposed are mutually contradictory, and some experience is necessary before one can obtain the most suitable size of cable to use, and satisfy the one desideratum in all transmission problems, i. e. *minimum all-round cost.* Thus, in designing the conducting part of the line of an electrical transmission and distribution scheme, minimum all-round cost implies (1) *minimum capital outlay*, and (2) *minimum waste* in the transmission. To economize capital outlay the weight of the metal forming the conductors should be as small as possible. This means that the conductors should have as small a cross-sectional area as possible; whilst to economize the losses in the line the conductors should introduce as little resistance as possible, which means that the cross-sectional area should be large.

As is well known, the dissipation of energy on account of resistance is accompanied by a rise of temperature, and it is of vital importance to avoid the production of an abnormal amount of heat, since increase of temperature results in increased resistance, injury to the insulation, and possible danger from fire. We may therefore conclude that if the conductors are too small, a large proportion of energy may be wasted, and that any saving in the capital outlay of the conductor which would result will be expended in fuel and machinery; consequently the determination of the best area to give to the conductors is one that requires much thought and attention. Much, however, can be done by applying the fundamental principles of the electric circuit which have already been explained. The problems met with in practice vary considerably according to the data given and to the conditions fixed, and the variable quantities which enter into the calculations relative to electrical conductors are:—

(1) The dimensions of the conductor and the resistance.

(2) The current to be carried.

(3) The current density.

(4) The loss or dissipation of energy.

(5) The drop or loss of electrical pressure in the circuit, or the drop in volts per unit length of the circuit.

These quantities are connected together in a very simple manner, and the solution of the problem depends upon the application of Ohm's law, Joule's law, and the law connecting the resistance of a conductor with its dimensions.

The resistance of a conductor may be expressed as follows:—

$$
\begin{aligned}
\mathrm{R}\ (\text{ohms}) &= \frac{0{\cdot}66}{1000000} \times \frac{l'' \ (\text{inches})}{\mathrm{A}\ (\text{square inches})} \\
&= \frac{0{\cdot}66}{10^6} \times \frac{4 \times 7}{22} \times \frac{l''\ (\text{inches})}{d^2\ (\text{inches})} \\
&= \frac{0{\cdot}84}{1000000} \times \frac{l''}{d^2} \quad \ldots \ldots \ldots \quad (58)
\end{aligned}
$$

when l'' and d are given in inches. The resistivity of commercial copper is 0·66 microhm or $\frac{0{\cdot}66}{1000000}$ ohm per cu. inch. If we use the mil-foot system the introduction of π in the calculation is avoided. The resistance of 1 foot of wire 1 mil in diameter is 10·81 ohms at 70° F. Therefore

$$
\mathrm{R}\ (\text{ohms}) = 10{\cdot}81 \times \frac{l'\ (\text{feet})}{\{d\ (\text{mils})\}^2} = \frac{10{\cdot}81 \times l'}{d^2\ (\text{cir. mils})} \quad \ldots \quad (59)
$$

Let us suppose that a large quantity of electrical energy has to be transmitted for power and lighting purposes, and that E is the E.M.F. at the source and E_1 the voltage at the distributing point. Then, if C be the current, the rate at which energy is supplied by the generator is EC watts, whilst $\mathrm{E}_1\mathrm{C}$ is the measure of the energy at the distributing point; $\mathrm{E} - \mathrm{E}_1 = e$ is the drop in volts in the feeders, and is the value of the electric pressure required to send the C amperes through the conductor of R ohms resistance. Therefore

$$
e = \mathrm{CR},
$$

and the energy wasted is $(\mathrm{E} - \mathrm{E}_1)\,\mathrm{C} = e\,\mathrm{C} = \mathrm{C}^2\mathrm{R}$ watts.

By combining the various relationships we may obtain working expressions for

(1) The area A, or diameter d, of a conductor which shall have a drop of e volts when carrying a current of C amperes, or which will absorb or dissipate w watts per second. In some cases it is an

advantage to use the drop in volts and absorption of energy per unit length of the conductor.

(2) The current or energy wasted in a conductor of given length, area, or diameter when transmitting a certain amount of energy P, with a loss of energy w or drop in volts e.

From equation (58) we have

$$d^2 = \frac{0{\cdot}84\, l''}{10^6\, \mathrm{R}}\,; \quad \mathrm{A} = \frac{0{\cdot}66\, l''}{10^6\, \mathrm{R}}\,; \quad \mathrm{R} = \frac{e}{\mathrm{C}},$$

$$\therefore \quad d^2 = \frac{0{\cdot}84\, l''\, \mathrm{C}}{10^6\, e} = \frac{0{\cdot}84\, \mathrm{C}}{10^6\, \dfrac{e}{l''}}$$

$$= \frac{0{\cdot}84\, \mathrm{C}}{10^6 \times \text{drop in volts per unit length}} \quad . \quad . \quad (60)$$

also

$$\mathrm{A} = \frac{0{\cdot}66\, l''\, \mathrm{C}}{10^6 \times e} = \frac{0{\cdot}66\, \mathrm{C}}{10^6 \times \dfrac{e}{l''}}$$

$$= \frac{0{\cdot}66\, \mathrm{C}}{10^6 \times \text{drop in volts per unit length}} \quad . \quad . \quad (61)$$

If P be the energy delivered at a pressure E_1 at the distributing point

$$\mathrm{C} = \frac{\mathrm{P}}{\mathrm{E}_1},$$

and if the percentage loss of energy in the line is p, then loss

$$w = \frac{p}{100} \cdot \mathrm{P} \quad \text{and} \quad e = \frac{p}{100}\, \mathrm{E}_1,$$

and by substitution

$$d^2 = \frac{0{\cdot}84 \times l'' \times \mathrm{P}}{10^6 \times \dfrac{p}{100} \times \mathrm{E}_1^{\,2}} = \frac{0{\cdot}84}{10^4} \times \frac{l''\, \mathrm{P}}{p\, \mathrm{E}_1^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Similarly

$$\mathrm{A} = \frac{0{\cdot}66 \times l'' \times \mathrm{P}}{10^6 \times \dfrac{p}{100} \times \mathrm{E}_1^{\,2}} = \frac{0{\cdot}66}{10^4} + \frac{l''\, \mathrm{P}}{p\, \mathrm{E}_1^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Using mil-foot units

$$\mathrm{A}\ (\text{cir. mils}) = 10{\cdot}81\, \frac{l'\, \mathrm{P}}{\dfrac{p}{100}\, \mathrm{E}_1^{\,2}} = 1081\, \frac{l'\, \mathrm{P}}{p\, \mathrm{E}_1^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

If the length of transmission, L feet $\left(\text{where } \mathrm{L} = \dfrac{l'}{2}\right)$, be taken instead of the total length of conductor (lead and return), this formula becomes

$$\mathrm{A}\ (\text{cir. mils}) = 2160\, \frac{\mathrm{L}\, \mathrm{P}}{p\, \mathrm{E}_1^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

a useful expression much used in American practice. The greatest length of transmission for a given conductor with a given drop and current or power transmitted is given by

$$L = \frac{1}{2160} \cdot \frac{A p E_1^2}{P} = \frac{1}{2160} \cdot \frac{A p E_1}{C}$$

$$= \frac{1}{21 \cdot 60} \frac{A e}{C} \text{ feet} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The heating effects have already been considered in Section II, Chapter II, so that we need not do more than insert here the following rules relating to the carrying capacities of wires.

Kennelly's Rule $C = 560 \sqrt{d^3}$,
where d = diameter of the core in inches.

Institution of Electrical Engineers' Rule for maximum current allowable in any situation

$$\text{Log } C = 0{\cdot}82 \log A + 0{\cdot}415$$

or

$$C = 2{\cdot}6\ A^{0{\cdot}82},$$

where A = area of conductor in 1000ths of a square inch.

Again, the ratio $\frac{C}{A}$ is the current density, and it is a common practice in house wiring to adopt a certain current density, i. e. so many amperes per square inch, as a means of determining the carrying capacity of wires. This method is simple, but its application is limited and should only be followed as a guide for giving the size of wires to use for short lengths of house wiring. Usually the rules issued by the fire insurance companies have to be satisfied in the case of house wiring, and a very common rule recommended by many fire offices is a current density of 1000 amperes per square inch of cross-sectional area. But, remembering that the heating effect is proportional to the square of the current, and that the current, according to this rule, is proportional to the area of cross-section, which increases as the square of the diameter, whilst the heat radiated is proportional to the surface of the conductor which is proportional directly to the diameter, it is clear that as the wire increases in diameter the temperature will increase at a rapid rate, and that the 1000-ampere rule is not safe for large conductors. In the case of small wires, again, more current than the rule allows may be carried, and an unnecessary large margin is allowed. The insurance companies recognize this, and the Phoenix Fire Office state in their rules that 'for currents up to 100 amperes, a current density of 1000 amperes per square inch may be employed, but for larger currents a less current density must be used. In any

case, the conductors must be absolutely safe with regard to heating, even if the current should become doubled.'

Another defect of this method is that the permissible drop of potential in the conductors is ignored, since the drop in volts is constant, and is approximately $2\frac{1}{2}$ volts per 100 yards of conductor. A value of the current density for a given length of cable and drop in volts may be obtained from equation (61), thus :—

$$\frac{C}{A} = \frac{10^6 \times e}{0{\cdot}66 \times l'} = 1515151{\cdot}5\,\frac{e}{l'}$$

$$= 126{\cdot}26\,\frac{e}{L_{1000}}. \quad (L_{1000} = \text{number of 1000 feet.})$$

§ 33. **House Wiring.** In the case of house wiring it is often necessary to introduce the number of lamps and the percentage drop in volts; and although wiring-tables facilitate the calculations and the determination of the size of wire to use, it is essential that local circumstances and each case should be considered separately, so that the lamps may receive the proper voltage and the leads may be designed so as to secure economy. If R is the resistance of the line, r is the resistance (hot) per lamp and n the number of lamps, then with a percentage drop, p, in the line we have

$$\frac{p}{100-p} = \frac{R}{\frac{r}{n}} \quad \text{and} \quad R = \frac{r}{n} \times \frac{p}{100-p}$$

but $R = \dfrac{10{\cdot}81\,l'}{d^2}$, if l' = length of lead and return in feet

and d = diameter in mils.

$$\left.\begin{aligned} \therefore\ d^2 &= \frac{10{\cdot}81\,l'\,n}{r} \times \frac{100-p}{p} \\ &= \frac{21{\cdot}6\,L\,n}{r} \times \frac{100-p}{p} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where $L = \frac{1}{2}\,l'$ = distance of transmission in feet.

The product L n is of the nature of a constant for a given installation, and is termed 'the lamp-feet' of the installation. r is also a constant for a given voltage lamp, and may be taken as $166\frac{1}{3}$ ohms for 100-volt lamps, 220 for 110-volt lamps, and $666\frac{2}{3}$ for 200-volt lamps. We may thus form a wiring table as follows. For 100-volt lamps and a 2 per cent. drop we have

$$d^2 = \frac{21{\cdot}6}{166\frac{2}{3}} \times L\,n \times \frac{100-2}{2}$$

$$= 6{\cdot}35\,L\,n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

It will be noticed that the size of a wire is given in terms of d^2, or

circular mils, and as the areas of the various wires are given in circular mils on page 360, it is an easy matter to pick out the gauge of the wire to use from the table. The following values refer to 100-volt lamps with the percentage drop from 1 to 10 per cent.

$$\begin{aligned}
d^2 &= 12{\cdot}83\ \mathrm{L}n \text{ for } 1 \text{ per cent. drop.} \\
&= 8{\cdot}51\ \mathrm{L}n \ \text{,,}\ 1\tfrac{1}{2} \ \text{,,} \ \text{,,} \\
&= 6{\cdot}35\ \mathrm{L}n \ \text{,,}\ 2 \ \text{,,} \ \text{,,} \\
&= 5{\cdot}06\ \mathrm{L}n \ \text{,,}\ 2\tfrac{1}{2} \ \text{,,} \ \text{,,} \\
&= 4{\cdot}19\ \mathrm{L}n \ \text{,,}\ 3 \ \text{,,} \ \text{,,} \\
&= 3{\cdot}57\ \mathrm{L}n \ \text{,,}\ 3\tfrac{1}{2} \ \text{,,} \ \text{,,} \\
&= 3{\cdot}11\ \mathrm{L}n \ \text{,,}\ 4 \ \text{,,} \ \text{,,} \\
&= 2{\cdot}46\ \mathrm{L}n \ \text{,,}\ 5 \ \text{,,} \ \text{,,} \\
&= 2{\cdot}03\ \mathrm{L}n \ \text{,,}\ 6 \ \text{,,} \ \text{,,} \\
&= 1{\cdot}72\ \mathrm{L}n \ \text{,,}\ 7 \ \text{,,} \ \text{,,} \\
&= 1{\cdot}49\ \mathrm{L}n \ \text{,,}\ 8 \ \text{,,} \ \text{,,} \\
&= 1{\cdot}31\ \mathrm{L}n \ \text{,,}\ 9 \ \text{,,} \ \text{,,} \\
&= 1{\cdot}16\ \mathrm{L}n \ \text{,,}\ 10 \ \text{,,} \ \text{,,}
\end{aligned}$$

Another very simple and useful rule, known as Sayer's wiring equation, may be used. If K = feet of cable per ohm resistance, then,

$$\text{since} \quad e = \mathrm{CR} \quad \text{and} \quad \mathrm{R} = \frac{l'}{\mathrm{K}},$$

$$e = \frac{\mathrm{C}\,l'}{\mathrm{K}} \quad \text{and} \quad \mathrm{K} = \frac{\mathrm{C}\,l'}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and K is of the nature of a constant for each gauge of wire, which may be calculated from the table on page 360; therefore the wire to use is the one whose value of K is nearest to the value of $\dfrac{\mathrm{C}\,l'}{e}$.

§ 34. **Economy of Conductors.** The economical design of a system of conductors forming a network for the transmission and distribution of electrical energy is one of the most important, and, at the same time, most difficult problems encountered by the electrical engineer. Many of the conditions imposed and the factors which have to be considered are mutually so conflicting that it is difficult to obtain both a high degree of efficiency and of economy. If a high efficiency be secured, it is necessary to use so much copper in the line that the first cost and the interest thereon may be so extravagant that from the economical point of view electrical transmission would be undesirable. If, on the contrary, the weight of copper of the line is reduced, the first cost of interest thereon will be reduced, but the efficiency will be less, and less power will be transmitted. In fact,

no hard and fast rule can be given which will apply in all cases, but there is, for each case, a certain cross-sectional area or weight of copper for the line which will give the greatest economy for the conditions imposed. Lord Kelvin first indicated (in 1881) the relationship which must hold if the capital outlay on the conductor varies in strict proportion to the weight of the metal, to obtain the most economical size of conductor. Lord Kelvin's law may be stated in words as follows:—

'The greatest economy is attained when the *sum* of the cost of the energy wasted and the interest on the capital expended *is a minimum.*'

To prove this we may proceed as follows:—

Let R = resistance of the conductor l miles long.
 C = current in amperes.
 f = fraction of capital outlay charged annually for interest and depreciation.
 p = price in £ per ton of copper in the form of a cable.
 m = tons of copper per mile of conductor per square inch section.
 A = area of cross-section of conductor in square inches.
 n = cost in pence per B.T.U. delivered to the conductor.
 h = number of hours per day that the current flows.

Then cost of energy wasted per annum is a (say), and

$$a = \frac{C^2 R \times 365\, hn}{1000} = 0{\cdot}365\, C^2 R\, hn,$$

but $$R = \frac{\sigma l}{A}, \quad \therefore \quad a = \frac{0{\cdot}365\, C^2 \sigma\, lhn}{A} \text{ pence.}$$

Interest on capital outlay charged annually is β (say), and

$$\beta = ml\, A \times p \times f \text{ pounds sterling} = 240\, ml\, A\, pf \text{ pence.}$$

And total annual expenditure is $a + \beta$, and

$$a + \beta = 240\, ml\, A\, pf + \frac{0{\cdot}365\, C^2 \sigma\, lhn}{A}.$$

It is required to find A so that the annual expenditure may be a minimum, and by the calculus it may be shown that *the sum of two quantities is a minimum when the two quantities are equal, provided that the product of the two quantities is a constant.* Obviously $a \times \beta$ is a constant, since A is the only variable, therefore for $a + \beta$ to be a minimum a must equal β, or

$$240\, ml\, A\, pf = \frac{0{\cdot}365\, C^2 \sigma\, lhn}{A}$$

from which $$A^2 = \frac{0{\cdot}365\, C^2 \sigma\, lhn}{240\, mlfp} = \frac{0{\cdot}365\, C^2 \sigma\, hn}{240\, mfp}.$$

Since σ = resistance of a conductor 1 square inch in section and 1 mile long, and is 0·045 ohm for copper, 0·045 may be substituted for σ in the case of copper, and taking $m = 9{\cdot}1$, we get

$$A = C\sqrt{\frac{0{\cdot}365 \times 0{\cdot}045\,hn}{240 \times 9{\cdot}1\,fp}}$$

$$= {\cdot}00274 \times C\sqrt{\frac{hn}{fp}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

from which it is obvious that the area of cross-section is independent of the length of the line, but is proportional to the current.

$$\text{Also current density} = \frac{C}{A} = \frac{1}{0{\cdot}00274}\sqrt{\frac{fp}{hn}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Example.—Determine the area of cross-section of a feeder which has to carry 250 amperes for 2,000 hours per year if the cost per B.T.U. is one penny, and the interest be 10 per cent. The price per ton of copper being £200.

In this case $h = \frac{2000}{365}$

and $f = \frac{10}{100} = \frac{1}{10}$,

$$\therefore \quad A = {\cdot}00274 \times 250\sqrt{\frac{2000}{365} \times \frac{1}{1/10} \times \frac{1}{200}}$$

$$= 0{\cdot}359 \text{ square inch.}$$

The current density is therefore

$$\frac{C}{A} = \frac{250}{{\cdot}359} = 696 \text{ or } 700 \text{ amperes per square inch.}$$

§ 35. **Three-wire system.** From what has already been said it is clear (1) that smaller conductors may be employed for distributing electrical energy, and consequently considerable economy may thus be effected by using high voltages for distributing purposes; (2) that independence of the individual lamps is secured by means of parallelism; and (3) that the difficulty encountered by the electrical engineer for lighting large buildings or institutions and extensive areas is that of increasing the range of distribution, i. e. the permissible distance between the generator and the lamps without undue loss or variation of pressure, or violating the laws of economy of conductors. One of the solutions of the problem of economical distribution of electrical energy consists of an extension of the simple parallel system and a modification of the multiple series system, forming what is known as the *three-wire system.* This system was invented simultaneously by Dr. Hopkinson in this country and by Edison in America, and, as proved by the numerous cases in which it has been adopted, and as we shall presently see,

it is an economical method of distributing electricity for lighting purposes. As may be inferred from its name, there are in this system three conducting mains, the centre one of which is a neutral or balancing wire, and, unlike its fellow in the multiple series system, it is connected to an intermediate point between the two dynamos so as to be in connexion with the positive terminal of one machine and the negative pole of the other. The arrangement is represented in Fig. 20, in which P and N are the positive and negative mains respectively, and M is the neutral or balancing wire. The source of electricity consists of two exactly similar dynamos connected in series, each giving the voltage required by an individual lamp, so that twice the pressure of the simple parallel system is used in the three-wire system. These dynamos, D_1 and D_2 in the figure, are shunt or compound machines, and it will be observed that the lamps are connected in groups of two in series, one on each side of M.

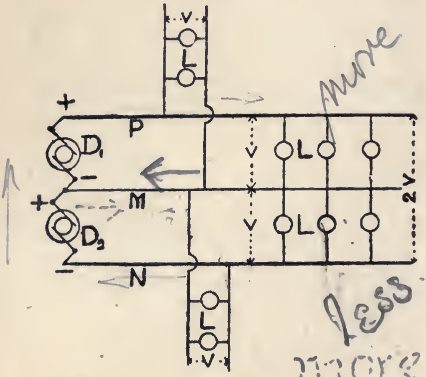


Fig. 20.

The essential feature of this system is that double the lamp voltage is maintained between the positive and negative mains, whilst the usual pressure is used for the service or house mains, which are connected between the middle wire and either the positive or negative mains, as shown in the figure. It is thus clear that the system is equivalent to two distinct parallel systems, the middle wire of the three-wire system playing the same function as the return of the first of the two parallel systems, and at the same time forming the lead of the second of the two systems. In other words, the part of the middle wire between the source and the first lamp is traversed by a current equal to the difference between the currents required by the lamps on the two sides of the middle wire.

As we have already stated, one of the advantages of using the three-wire system is the economy in the copper which results, and to decide what the gain really is, it is usual to compare the weights of the copper required for the conductors for a simple parallel system and a three-wire system supplying the same number of lamps. And as the statements made with reference to what the exact saving in copper is vary considerably, it is necessary to point out that two factors may enter into our calculation—i. e. (1) the

density of the current decided upon, and (2) the permissible drop in volts. Then, again, the middle wire may, or may not, be of the same section as the outside wires; and for lighting purposes it is obvious that the supply mains should be so arranged that the number of active lamps in use at any time on one side of the neutral wire is approximately equal to the number of lamps lit simultaneously on the other side.

By paying proper attention to this matter when designing the system, a considerable saving in the weight of copper may be effected with respect to the middle wire, for in practice it is found possible to distribute the supply mains so that not more than half the maximum current, passing along the positive or negative distributor at any time, traverses the neutral conductor. Very often it will be considerably less than half; consequently the area of the cross-section of the middle wire may, and often is, reduced to half that of the outside wires. Obviously the most care is required when the district lighted consists partly of business premises and partly of residences. For large installations, where the number of lamps working at any time are liable to vary, it has been found convenient and possible to introduce switching arrangements for transferring a portion of the load from one side of the middle wire to the other. By this means any installation may be subdivided into two halves at any time, and balance between the two loads readily secured, so that the middle wire never carries much current.

The section of the middle wire, however, should not be less than that of the outside wires when two motors are connected in series, and are supplied by means of three wires with energy from two dynamos, also connected in series. When both motors are working, the middle wire—connected between the common junction of the generators and that of the motors—plays the part of a balancing wire, as in lighting circuits. Should, however, only one motor be at work, then one of the outer wires is passive, and the middle wire has to carry the same current as the active outside wire. In cases where several motors are grouped together, after the manner of lamps, then, of course, the same considerations apply as we have already mentioned in connexion with lighting circuits.

Whilst considering the middle wire, we may, in passing, refer to the magnitude and direction of the current passing along it. Let C_p, C_n, and C_m be respectively the currents traversing the conductors, P, N, and M of Fig. 20 at any particular time; then, if the same number of lamps are in use on both sides of the middle wire, no current passes along the neutral wire M, either from or to the dynamo, since

$$C_p = C_n \text{ and } C_m = 0.$$

It is for this reason that the middle wire is termed the *neutral* wire. If more lamps are in use between P and M than between M and N

$$C_p \text{ is greater than } C_n, \text{ and } C_m = C_p - C_n$$

that is, a current equal to C_m passes along the middle wire from the lamps to the dynamo D_1. On the contrary, if

$$C_n \text{ is greater than } C_p, \text{ then } C_m = C_n - C_p,$$

and a current equal to C_m passes along the middle wire from the dynamo D_2 to the lamps. If $C_n = 0$, then $C_m = C_p$, and so on. It thus follows that each dynamo supplies the energy required individually for the corresponding groups of lamps. It is thus evident why the middle wire is also termed the *balancing* wire, and that the middle wire plays the part of a positive, neutral, or negative main at different times according to the equality or inequality of the two loads on the two sides of the middle wire. In fact, the third wire was introduced and connected to the generators because of the difficulty of maintaining an exact balance between the number of lamps on the two sides of the middle wire, without introducing special arrangements.

In Fig. 21 a special case is represented, from which it will be observed that we have an interesting arrangement indicating an equal number of active lamps on the two sides of M, and that

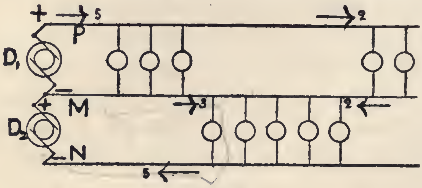


Fig. 21.

although no current passes from, or to, the dynamos along the neutral wire, there are currents of different intensity and direction traversing different parts of that conductor, as indicated by the arrow-heads. By assuming that each lamp takes one ampere, we are able to give the magnitudes of the currents along with the directions for each conductor.

The three-wire system is used considerably for distributing purposes and for large installations, because it offers the following advantages over the simple parallel or two-wire system.

(*a*) It enables the E. M. F. to be maintained in a system of conductors for electrical supply at a fixed and constant potential.

(*b*) It combines the advantages of higher pressures for distribution

in the distributing mains, and the easy manipulation of the simple parallel system by adopting lower and safe pressures in the service mains. It is clear that more energy may be delivered because of the higher pressure used, and that larger areas may be served without increasing the losses in the mains.

(*c*) It permits of the independence of each lamp. Although the lamps are connected two in series on a parallel system, the failure of lamps on one side of the neutral wire does not affect the lamps on the other side; consequently, any lamp may be turned out independently of all the others.

(*d*) It secures a considerable reduction in the cost of the copper in the distributors consistently with efficiency and safety.

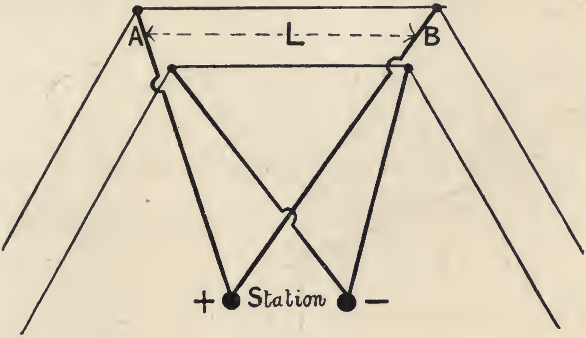


FIG. 22.

Worked Examples. (1) Determine the diameter of a conductor which is required to transmit and deliver 10 kilowatts of electrical energy at 100 volts to a point 275 yards from the generator, if the loss of energy in the line has not to exceed $7\frac{1}{2}$ per cent. of the power delivered.

$$10 \text{ Kw} = 10000 \text{ watts},$$

$$\therefore \quad C = \frac{10000}{100} = 100 \text{ amperes}.$$

A loss of 7·5 per cent. in the line gives a drop in volts of $\frac{7 \cdot 5}{100}$ of 100, or 7·5 volts along the 550 yards of the conductor.

$$d^2 = \frac{0 \cdot 84}{10^6} \times \frac{l'' C}{e}$$

$$= \frac{0 \cdot 84}{10^6} \times \frac{550 \times 36 \times 100}{7 \cdot 5}$$

and $\quad d = 0 \cdot 471$ inch.

(2) Determine the size of the distributors which supply current at the rate of 0·3 ampere per foot if the distance between the feeding-points is 200 yards, and the maximum difference of potential between any two lamps does not exceed 5 volts.

In Fig. 22 let A and B be the feeding-points, and L feet the distance between A and B. In such a case as this, where the distributing system is

fed at both ends, and the distribution is perfectly symmetrical, it is an easy matter to find the point of lowest pressure. In all cases the point of lowest pressure is the electrical centre (centre of gravity, as it were) of the distributing main, and the drop in volts to this point from one feeding-point must be equal to that from the other feeding-point. And since drop in volts is

$$e = \mathrm{CR} = \mathrm{C}lr,$$

where l is the distance of lowest pressure from a feeding-point, and r the resistance of unit length, it is obvious that

$$\Sigma(\mathrm{C}_1 l_1) = \Sigma(\mathrm{C}_2 l_2),$$

where $\Sigma(\mathrm{C}_1 l_1)$ and $\Sigma(\mathrm{C}_2 l_2)$ represent the summation of the ampere-feet (products of currents and distances) on the two sides of the point of lowest pressure. Where the current is taken off at regular intervals and by equal amounts the electrical centre is midway between the feeding-points, and $\mathrm{C}_1 l_1 = \mathrm{C}_2 l_2$. Again, $\mathrm{C}_1 = \mathrm{C}_2 = \frac{\mathrm{C}}{2}$, but the mean current, $\frac{\mathrm{C}_1}{2}$ or $\frac{\mathrm{C}}{4}$, is that which is a measure of the drop. The total current required is

$$\mathrm{C} = c\mathrm{L} = \cdot 3\mathrm{L}, \quad \text{and} \quad \mathrm{C}_1 = \mathrm{C}_2 = \frac{1}{2} \times \cdot 3\mathrm{L} = \frac{1}{2} \times 200 \times 3 \times \cdot 3$$

$$\text{Drop in volts} = \frac{\mathrm{RC}_1}{2} = \frac{\mathrm{L}\sigma}{\mathrm{A}} \times \frac{c\mathrm{L}}{4} = \frac{1}{4}\sigma c \frac{\mathrm{L}^2}{\mathrm{A}}$$

The resistance of the mains between the feeding-point and the electrical centre (lead and return) is $\frac{\cdot 66}{10^6}\frac{\mathrm{L}}{\mathrm{A}}$

or
$$\mathrm{R} = \frac{\cdot 66}{10^6} \times \frac{200 \times 36}{\mathrm{A}}$$

$$\therefore \text{ Drop in volts} = \mathrm{RC}_1 = \frac{1}{4} \times \cdot 3\mathrm{L} \times \frac{\cdot 66}{10^6} \times \frac{\mathrm{L}}{\mathrm{A}}$$

and
$$5 = \frac{\cdot 66}{10^6} \times \frac{7200}{\mathrm{A}} \times \frac{1}{4} \times 200 \times 3 \times \cdot 3$$

$$\therefore \mathrm{A} = \frac{\cdot 66 \times 7200 \times 180}{4 \times 5 \times 10^6}$$

$$= 0{\cdot}042768 \text{ sq. inch.}$$

(3) A, B, and C are three points in a straight line 100 yards apart. At A is a generator, at B a group of 200 lamps taking 0·25 ampere at 220 volts, whilst at C there are 150 200-volt lamps, taking 0·3 ampere each. Determine the diameter of the cable to use between A and B (lead and return) and between B and C (lead and return) if the pressure at the generator is 235 volts.

It is obvious that the following relations exist:

$$\text{Drop in volts between B and C} = e_2 = 2\mathrm{C}_2\mathrm{L}_2 r$$

and ,, ,, ,, A and B $= e_1 = 2\,\mathrm{L}_1 r(\mathrm{C}_1 + \mathrm{C}_2)$

where r is resistance per unit length, and L = distance between two points.

$$\mathrm{C}_1 = 200 \times 0{\cdot}25 = 50 \text{ amperes}$$
$$\mathrm{C}_2 = 150 \times 0{\cdot}3 = 45 \text{ amperes}$$

Therefore for cable between A and B we have

$$e_1 = 235 - 220 = 15 \text{ volts} = (\mathrm{C}_1 + \mathrm{C}_2)\,2\,\mathrm{L}_1 r$$

$$= (50 + 45)\,2 \times 100 \times 3 \times \frac{10{\cdot}81}{d^2}$$

$$\therefore\ d^2 = \frac{200 \times 3 \times 10{\cdot}81 \times 95}{15} = 41078 \text{ cir. mils}$$

and $\qquad d = 202{\cdot}6 \text{ mils} = 0{\cdot}2026 \text{ inch.}$

For cable between B and C we have

$$e_2 = 220 - 200 = 20 \text{ volts}$$
$$= 2 \times 100 \times 3 \times \frac{10{\cdot}81}{d^2} \times 45$$
$$\therefore\ d^2 = \frac{200 \times 3 \times 10{\cdot}81 \times 45}{20} = 14593{\cdot}5 \text{ cir. mils}$$

and $\qquad d = 120{\cdot}8 \text{ mils} = 0{\cdot}1208 \text{ inch.}$

(4) Current is required along a street 1000 yards long at the rate of 0·5 ampere per yard of frontage. Determine the sectional area of the cable so that the difference of pressure between any two lamps shall not exceed 4 volts in the following two cases :—

(*a*) The current being supplied at one end.

(*b*) The current being supplied midway.

Total current $= 0{\cdot}5 \times 1000 = 500$ amperes; and since the current is uniformly distributed along the mains, the drop in volts between the first and last lamps will be equal to the product of the *mean current* into the resistance between the first and last lamp; therefore, in such cases

$$e = \frac{C}{2} \times 2L \times r.$$

(*a*) Current supplied at one end.

$$e = \frac{C}{2} \times 2L \times r$$

Here $e = 4$ volts, $C = 500$ amperes, $L = 1000$ yards, and $r = \frac{10{\cdot}81}{d^2}$, where d = diameter in mils.

$$\therefore\ 4 = \frac{500}{2} \times 2 \times 1000 \times 3 \times \frac{10{\cdot}81}{d^2}$$

and $\qquad d^2 = 4053750 \text{ cir. mils.}$

$$\therefore\ d = 2013 \text{ mils} = 2{\cdot}013 \text{ inches.}$$

(*b*) Current supplied midway.

$$\text{Total current for each half of street} = {\cdot}5 \times 500$$
$$= 250 \text{ amperes,}$$
$$\text{and mean current} = \frac{250}{2} \text{ amperes.}$$
$$\therefore\ 4 = \frac{250}{2} \times 2 \times 500 \times 3 \times \frac{10{\cdot}81}{d^2}$$

and $\qquad d^2 = 1013437{\cdot}5 \text{ cir. mils.}$

$$\therefore\ d = 1006 \text{ mils} = 1{\cdot}006 \text{ inch.}$$

EXERCISES III.

Electric Light and Power Circuits.

(1) A dynamo is 100 yards from a house, the conductor has a resistance of 0·002 ohm per yard, and there are 150 30-watt 100-volt lamps to be fed. What E.M.F. must the dynamo give? (C. and G.)

(2) Two lamps of 100 and 150 ohms each, when running, are put in parallel with each other, and the pair are put in series with a lamp of 100 ohms. What E.M.F. will be needed on the system in order that it may consume 250 watts? (C. and G.)

(3) Calculate the horse-power required for an installation of twenty-five 60-watt 100-volt lamps connected in parallel, if the leads have a resistance of 0·2 ohm, and the branch leads a resistance of 0·01 ohm for each lamp, and assuming that the engine and dynamo have an efficiency of 60 per cent.

(4) Thirty accumulators, each having an E.M.F. of 2·1 volts and a resistance of 0·002 ohm, are employed to feed incandescent lamps. If the lamps require 45 volts and 1·25 amperes each, what is the maximum number of lamps that can be employed? (C. and G.)

(5) Edison glow-lamps requiring 108 volts potential difference, and 0·72 ampere each, are required to be fed by accumulators each having 2·1 volts E.M.F. and 0·0017 ohm resistance. What is the least number of such accumulators arranged in series that must be employed to feed 200 of these lamps in parallel? (C. and G.)

(6) A 110-volt 16-c.p. glow-lamp takes 0·5 ampere. How many of these lamps, connected in parallel, can be run from a battery of accumulators of 55 cells in series, the E.M.F. and internal resistance of each being 2·25 volts and 0·002 ohm respectively, if $1\frac{2}{3}$ volts are lost in the mains?

(7) A group of twenty-five 16-c.p. 100-volt lamps, each taking 0·55 ampere, are to be run by a dynamo 50 yards away. What must be the cross-section of the cable, if the drop is not to exceed $\frac{1}{2}$ a volt? The resistance of a cubic inch of copper may be taken as 0·66 microhm. (C. and G.)

(8) In a water-power plant, the dynamo which produces a fixed P.D. between its terminals of 120 volts is 300 yards away from the house. The usual load consists of 200 100-volt 55-watt glow-lamps. What size leads should be employed if the resistance of a cubic inch of copper be 0·66 microhm? (C. and G.)

(9) A dynamo, maintaining a constant pressure of 220 volts between its terminals, supplies a power of 18000 watts to a house 200 yards away. What must be the cross-section of the copper of the leads so that not more than 4 per cent. of the power may be wasted in them? Resistance of a cubic inch of copper may be taken as 0·66 microhm. (C. and G.)

(10) A compound-wound dynamo producing a terminal P.D. of 150 volts is used to charge sixty storage cells each having an E.M.F. of 2·2 volts and a resistance of 0·001 ohm. If the leads joining the

dynamo and cells have a resistance of 0·2 ohm, what will be the current generated? (C. and G.)

(11) The declared pressure at which current is supplied to houses by mains coming from a central station is raised from 100 to 220 volts. If the percentage loss in the mains is to remain the same as before, by how much per cent. will their carrying capacity be increased when the heating limit has not to be regarded? (C. and G.)

(12) A dynamo, driven with water-power at a constant speed, supplies energy to fifty 60-watt lamps and a 20-B.H.P. motor, used for driving a saw-mill 1000 yards away; consider what kind of dynamo and motor you would use, and what electric pressure would be suitable. Calculate approximately the cross-section you would give to the mains, the drop in pressure that would occur, and the power required to drive the dynamo at full load. (C. and G.)

(13) A dynamo supplying 250 volts at its terminals feeds 500 glow-lamps in parallel, each taking 66 watts. The drop in the mains is 12 per cent. of dynamo voltage. The resistance of the dynamo is 0·025 ohm; determine the total current, the lost volts in the dynamo, and the electrical efficiency of the installation.

(14) Determine the diameter of a cable required to feed a 25-H.P. motor working with an efficiency of 90 per cent. and taking current at 400 volts. The E.M.F. of the generator is 440 volts, and the distance between the generator and motor is 100 yards.

(15) Determine the number of 100-volt lamps, each taking 0·6 ampere, which can be fed by a 7/16 cable so that the drop in volts does not exceed 5 volts when the distance between the generator and the lamps is 50 yards (1) by Sayer's rule, and (2) by the lamp-feet rule. The equivalent diameter of a 7/16 cable may be taken as 170 mils and area 0·0229 sq. inch.

(16) What current-density corresponds to the two cases of No. 15?

(17) A system of distributors is fed from both ends, the distance between the feeding-points being 407 yards, and the demand for current is at the rate of 0·5 ampere per yard. Determine the area of cross-section of the distributing mains so that the maximum difference of potential between any two lamps is 2 volts. 1 cubic inch has 0·66 microhm resistance.

(18) What is the maximum distance between two feeding-points if the distributors supply current at the rate of 0·3 ampere per foot, and the current-density adopted is 750 amperes per square inch? The area of cross-section of the distributor is 0·2 square inch. If no lamp receives less than 200 volts, determine the P.D. at the feeding-points and the maximum percentage drop in volts between any two lamps.

(19) Compare the cost of a three-wire system of distribution with

that of a two-wire system according to the following conditions, the load being the same in all cases :—

I. Same gauge and length of wire in the distributing mains.
II. Same gauge of mains and the same drop in volts.
III. Equal lengths of mains and the same current-density.
IV. Equal lengths of mains and the same percentage fall of pressure.

The cross-section of middle wire to be half that of outers.

(20) A building of four floors has ninety 8-candle-power lamps on each floor: the height of each floor is 18 feet, and the mains run straight up in the middle of the building. On each floor is a passage extending 100 feet each way from the middle, and in the passage on each side of the middle are five lamps. Opening out of the passage, on each floor, there are twenty rooms to the front and twenty rooms to the back, each room containing two lamps. Calculate the size of the mains going from the bottom to the top of the building, of the submains in the passages, and of the lamp leads, on the supposition that when all the lamps are turned on the drop in pressure from the basement to the farthest lamp does not exceed 2 volts, and that the pressure supplied is 100 volts. (C. and G.)

(21) A four storied mill is to be lighted with 200 16-candle-power 200-volt lamps on each floor, and the height of the mill is 48 feet. Design a system of mains from the switchboards on the respective floors to the main switchboard—which is 80 yards distant from the point where the mains enter the building—such that each floor shall be independent of all the others, and the drop of pressure between any switchboard and the main switchboard shall not exceed $1\frac{1}{2}$ per cent. when all the lamps in the building are turned on. (C. and G.)

(22) A rectangular building consists of five stories of 12, 12, 11, 11, and 10 feet in height respectively, and each floor measures 50 feet by 100 feet, and is to be lit with eighty 16-c.p. lamps, the lamps being distributed evenly over the floors on the two-wire system, and regarded as taking 60 watts each. The main conductors pass up the wall at about the centre of one of the long sides of the building, and 3-ampere-way distributing fuse-boxes are to be provided on each floor; the lamp circuits are to be of 3/20 conductor, and may be run parallel with either wall, but not diagonally. Calculate the sizes of the mains, and the number of branch circuits, so that the maximum fall of potential when all the lamps are on does not exceed $1\frac{1}{2}$ per cent.—

(*a*) When the supply is at 100 volts pressure.

(*b*) When the supply is at 200 volts pressure.

The resistance of 3/20 conductor is 0·27 ohm per 100 feet, and of a cubic inch of copper 0·66 microhm. (C. and G.)

(23) 250 incandescent lamps are supplied with current from a dynamo 1000 yards away. Each lamp takes 0·6 ampere at 100 volts. Determine (*a*) Total lamp resistance.

(*b*) The area of the cable, the current-density being 750 amperes per square inch.

(*c*) The drop in volts in the cable, assuming that it absorbs $\frac{1}{20}$th of the power used in the lamps.

(*d*) The current-density if the cable be changed, so that the loss is 4 per cent. of the power used in the lamps.

(24) A pair of feeders, each half a mile long, have to deliver 100 kilowatts at 440 volts. What cross-section must they have so that the loss in them may not exceed 5 per cent. of the power delivered? (1 inch cube of copper has a resistance of $\frac{2}{3}$rd microhm.) (C. and G.)

(25) A district requires 5000 200-volt 50-watt lamps. Assuming that current for 500 lamps is taken from the mains at points 10 yards apart, and that the current-density is 750 amperes per square inch, determine the drop in volts between the distribution point and the first batch of lamps (at one end of mains), and also for the last batch of lamps: (1) the current being supplied at one end (first batch end), and (2) the current being supplied midway.

Also give the cross-sectional area in both cases, and the E.M.F.'s required at the distribution point.

(26) A two-wire distribution main runs down and ends at the bottom of a certain street, having a uniform load diagram of 1·5 ampere per yard. If the current-density is 1000 amperes per square inch and the pressure at the feeding-points is 102 volts, determine the cross-sectional area of the mains, so that the pressure at the bottom of the street does not fall below 98 volts at full load. The resistance of 1 mile of copper 1 square inch in section may be taken as 0·0455 ohm.

(27) A, B, and C are three points, each at a distance of 157 yards from the next. At A there is a dynamo; at B a group of 200 incandescent lamps requiring 110 volts and 0·75 ampere each; and at C a group of 250 lamps requiring 100 volts and 0·83 ampere each. Calculate the diameter of the copper conductor that must be used for the lead and return wires between A and B, and between B and C respectively, if the potential difference maintained at the terminals of the dynamo is 112 volts. 1 yard of commercial copper wire, 1 square inch in section, has 0·00002435 ohm resistance. (C. and G.)

(28) Current is required along a street 2000 feet long at the rate of

1 ampere per 10 feet of frontage. Give sectional area of conductor so that the difference of pressure between any two lamps shall not exceed 2 volts in the following two cases:—

(*a*) The current being supplied at one end.

(*b*) The currrent being supplied midway.

N.B.—The resistance of a copper bar 1000 feet long and 1 square inch in section may be taken at 0·008 ohm. (C. and G.)

(29) Current is required along a street 2000 feet long at the rate of half an ampere per foot run, or in all 1000 amperes. The voltage drop between any two points of the mains must not exceed 4 volts. The generating station is at one end of the street. State size of main and total weight of copper for the two following cases:—(*a*) The station is directly connected to the end of the main; (*b*) the station is connected to the middle of main by a feeder of such size that 10 volts is lost in the feeder. Include weight of feeder in giving the total copper weight. N.B.—A bar of copper 1 square inch in section and 1 mile long has a resistance of 0·0425 ohm. 1 cubic inch of copper weighs ·32 pounds. (C. and G.)

(30) You have in a street a pair of mains, each conductor 0·2 square inch in area (0·12 ohm per 1000 yards), and the demand for current is at the rate of one 30-watt 100-volt lamp per yard run. Give distance between the feeding-points, so that the maximum variations of pressure between any two consumers shall not exceed 3 volts. (C. and G.)

(31) You have to deliver 115 electrical horse-power by means of a continuous current $1\frac{1}{2}$ miles distant from the generator. The terminal pressure at the generator is 1000 volts, and the loss due to line resistance is assumed to be $7\frac{1}{2}$ per cent. of the power delivered. Give the size and weight of the conductor required. N.B.—A bar of copper 1 square inch in section and 1 mile long has a resistance of 0·0425 ohm. The specific weight of copper is 8·9. The insulation of the line is perfect. (C. and G.)

(32) To transmit 1000 H.P. of electrical energy a distance of 6 miles for electric lighting it is proposed to use an electrical pressure of 10000 volts. Calculate (*a*) the current, and (*b*) the resistance of the line per mile, on the condition that the permissible loss is 5 per cent. of that received.

(33) A room containing 60 lamps is at a distance of 200 yards from the dynamo; what must be the diameter of the copper main leads so that the potential difference at the lamps is not less than the potential difference at the terminals of the dynamo by more than 1 per cent. (*a*) when the lamps require 45 volts; (*b*) when they require

110 volts? 1 yard of commercial copper wire 1 square inch in each section has 0·00002435 ohm resistance. (C. and G.)

(34) The pressure at which a certain amount of power is delivered to a line is doubled. Determine (*a*) the effect on the power wasted, (*b*) how much the area of cross-section may be diminished so that the losses may be the same as before, and (*c*) how much the original line may be lengthened so that the losses may be the same as before.

(35) The electric lighting of a block of buildings is effected from an engine-room 1350 yards distant. The wiring of the block is arranged for the three-wire system, and is fed by a three-wire feeder, the cross-sectional areas of the conducting cores of which are 0·5, 0·25, 0·5 square inch respectively. There are 700 lamps of 16-c.p. in use on the circuits connected with one side of the three-wire system and 550 on the other. What voltages must be maintained at the generating station ends of the feeder between the positive and negative conductors and the neutral respectively, in order that the pressure between the terminals of each of the distributing circuits may be 205 volts? The current required by a 16-c.p. lamp may be taken as 0·3 ampere, and the resistance of a cubic inch of copper 0·66 microhm. (C. and G.)

(36) Determine the most economical current-density of a feeder which has to be used twelve hours per day, if the cost per B.T.U. is 0·8*d.*, price of copper per ton £200, and interest be charged at 9 per cent.

(37) In a certain transmission plant 400000 watts of electrical energy at 1000 volts are delivered at a point 440 yards from the generating station. If the conditions are the same as in No. 36, determine (*a*) the most economical cross-sectional area of the feeder; (*b*) the drop in volts per mile; and (*c*) the E.M.F. at the generating station.

(38) Determine (*a*) the energy wasted in watts; (*b*) the cost of the waste energy per annum; and (*c*) the interest on the capital expended, in the above examples (Nos. 36 and 37).

(39) Determine (*a*) the cross-sectional area of the feeder; (*b*) the energy wasted in watts; (*c*) the cost of the waste energy per annum; and (*d*) the interest on the capital expended, if the 400000 watts of electrical energy are delivered at 1000 volts 440 yards from the generating station at which the pressure is 1010 volts, other conditions remaining the same as in No. 36.

(40) Show that when the greatest drop in volts between the mains of a distributing system is e volts, the amount of copper in the mains is least when the cross-sectional area of the mains varies as the square root of the current in it.

(41) Show that when satisfying the conditions of greatest economy in distributing a power P the efficiency is $1-\frac{RP}{E^2}$, where E is the E.M.F. of the generator and R is the total resistance of the circuit.

(42) Show that if the permissible waste in transmission is P_1 when the useful load is P, the actual current-density to adopt is $a = \beta \times \frac{P_1}{P_2}$, where P_2 is the waste if the current-density were β.

(43) Illustrate Lord Kelvin's law of economic proportion in cost of lines by an example of continuous current power transmission; taking the case of a line for delivering 100 H.P. at a distance of 4 miles: maximum voltage 3000 volts. (C. and G.)

Give the cross-sectional area if cost of B.T.U. is 0·75*d.*, price of copper £200 per ton, interest 10 per cent., and that the power is required twelve hours per day.

CHAPTER IV

MAGNETISM

Section I. Elementary Principles of Magnetism.

§ 36. **Magnetism and Magnets.** The term *magnetism* is the name given to the influence in virtue of which certain bodies called magnets possess distinctive *attractive* and *directive* properties, as well as *magnetic polarity.*

Now, these properties are (as will be apparent when we consider the magnetic condition of a magnetized iron ring) in reality secondary effects, and it is important to notice that they imply the existence of force. Furthermore, each of these properties impresses us with the fact that magnetic force is transmitted across space; consequently there exists magnetic force in the neighbourhood of a magnet, and as a result a *field of force*, known as a magnetic field, is formed. Hence the following definition :—

A Magnetic Field is the region or space within which magnetic force manifests itself.

There are several ways of representing magnetic fields graphically; the simplest, and at the same time a very effective method, is as follows :—

To get the graphical representation of the field of a bar-magnet, sprinkle fine iron filings upon a sheet of white cartridge-paper (not too smooth) placed over the bar-magnet, and it will be found that the filings arrange themselves so as to form curious, yet beautiful, chains of filings in the shape of distinct curves connecting the two ends of the magnet. In a similar manner the magnetic figure of any magnetic field may be obtained. The field of a bar-magnet in one plane is shown in Fig. 23 A.

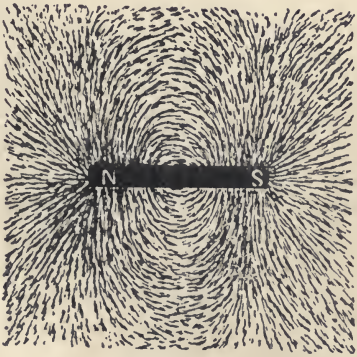


FIG. 23 A

These curves represent graphically the directions along which the

magnetic force acts, and they are extremely useful by giving an insight into the nature of magnetic fields. They depict, for instance, what a strange atmosphere of power surrounds a magnet, and we also learn that at every point in the field the magnetic force has a well-defined direction, and also a definite intensity. Since magnetic force proceeds along these definite paths or lines, magnetic force is √ a *vector quantity*, that is, it has direction as well as magnitude, and the paths traversed by magnetic force are termed *lines of force*. Faraday, who adopted this method of investigating magnetic phenomena, introduced the 'line of force theory,' a conception which assumes that a magnetic field is occupied with lines of force. These lines, of course, have no actual existence; still, they may be used to furnish us with data by means of which we are able to determine the magnetic force at any point of the field. A line of force is thus defined :—

√ *A line of magnetic force is an imaginary line in a magnetic field, which at every point is in the direction of the resultant magnetic force at that point.*

It will also be noticed from the figure that lines of force radiate from one pole of the magnet to the other, forming complete curves, which do not cut, cross, or merge into one another. Every curve upon the one side of a magnet has its counterpart on the other side. According to the modern conception of magnetic fields, action at a distance is due to a continuous transmission of a particular kind of stress through a medium which readily propagates this stress, consequently all magnetic phenomena and properties are accompanied by, if they are not actually the effects set up by, the condition of stress. This condition of stress gives rise to two states of strain: (1) a tension along the lines of force, and (2) a pressure across them. Faraday also conceived that the medium was under a strain, since he attached to lines of force the physical properties that the lines tend 'to shorten themselves from end to end,' and 'to repel one another as they lie side by side.'

The space forming a magnetic field is in a different physical condition to what it is when the magnetic force is absent; it is, in fact, *in a condition of strain*, and if a copper disc, supported in a plane perpendicular to the direction of the magnetic field by a piece of wire, be twisted about an axis perpendicular to the direction of the field, it will be found that there is considerable resistance to motion, and that it is far more difficult to rotate the disc when the magnetic force is present than when it is absent, as if the space were filled with some viscous liquid. Upon removing the magnetic force the opposition to motion disappears, indicating that the condition of strain accompanying a magnetic field has been removed.

As may be proved experimentally, magnetic force, due to magnets, appears to spring from two centres of force or points of maximum intensity, situated nearly at the ends of the magnet. Such points are called the *poles* of the magnet. The quantity of magnetism acting as if concentrated or accumulated at a pole is termed the *strength of the pole.* The lines of force of a magnet are partly external and partly internal (see Fig. 23 B), since the magnetization of a bar of iron or steel is continuous throughout its length. The external lines, or, briefly, the lines of force, emerge from the N or + pole, and enter the magnet again at the S or − pole, and borrowing terms from the analogous subject of fluid motion, the N pole of a magnet may be looked upon as a *source,* and the S pole as a *sink*; in other words, the N pole is a point of egress, and the S pole one of ingress. The internal lines, which lie within the magnet and

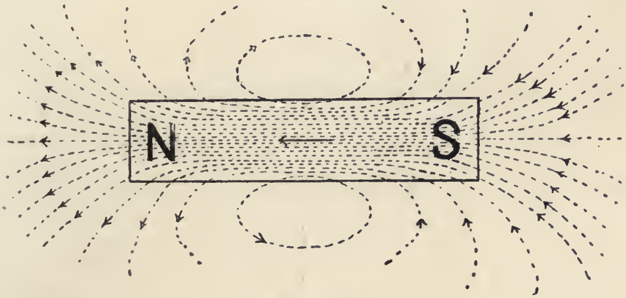


FIG. 23 B.

are not represented by the filings, are known as *lines of magnetization.* The *axis* or *polar line* of a magnet is the straight line joining the poles of a magnet. As we have to deal with quantitative relationships, it will be well here to define unit magnetic pole.

A centimetre-gramme-second (C.G.S.) unit magnetic pole is that N magnetic pole which exerts a force of 1 *dyne upon an equal magnetic pole* 1 *centimetre distant in air.* And a pole of strength m is equal to m unit poles.

The fundamental law of magnetism states that *like magnetic poles repel and unlike magnetic poles attract,* and the contrariety of properties (attraction and repulsion) exhibited by the two ends of a magnet is known as *magnetic polarity.* Coulomb was the first to investigate the law of force between two magnetic poles and to experimentally establish the truth of the law of inverse squares by direct measurements upon long thin bar-magnets with his torsion balance. Hence Coulomb's law, which states that *the magnetic force exerted between two*

poles is directly proportional to the product of the pole strengths, and inversely proportional to the square of the distance between them. Stated algebraically, this is

√

$$f \text{ (dynes)} = \frac{m \times m_1}{d^2} \text{ (in air)},$$

where m and m_1 are the pole strengths in C.G.S. units, and d the distance between the poles in centimetres.

Let us suppose that we have two very long and thin magnets equally magnetized (equal pole strengths), one of which is suspended vertically from one end of a very delicate balance with the N pole pointing downwards, equalizing weights w_1 grammes being placed in the opposite pan. If the second magnet be now placed vertically under the first magnet with its S pole adjacent to the N pole of the upper magnet, attraction results, which may be counterbalanced by adding weights in the pan. Let w_2 grammes be the additional weight when the beam is horizontal and the two unlike poles are 1 centimetre apart; then, since the weight of 1 gramme is equal to a force of 981 dynes, the attractive force exerted between the two poles is equal to 981 w_2 dynes. If $w_2 = \frac{1}{981}$ gramme (approximately a milligramme) unit force, the dyne will be exerted between the two poles; and since the poles are equal and the distance between them 1 centimetre, the pole strength of each magnet is, by Coulomb's law and definition, unity.

If the lower magnet has a pole strength of m units, then the attractive force will be m dynes (i. e. $\frac{m}{981}$ gramme will be additional weight in the pan), provided the distance apart remains 1 centimetre. If the distance between the poles be increased to d cms., then the force exerted will be $\frac{1}{d^2}$ times that for unit distance. It must be noticed that the above remarks are only true on the assumption that the magnets used are so long that the other poles have no influence upon the force exerted. This method of treating this part of the subject renders the following statement obvious :—Two poles are equal when they individually exert the same force on a third pole placed the same distance away.

As with other fields of force the properties of a magnetic field are completely known when the *direction of the lines of force* and the *intensity of the field* are specified. With respect to the former, the positive direction of a line of force is assumed to be that direction which would be taken by a free N pole (were it possible to obtain

one) placed in a magnetic field. Such a pole, if placed in a magnetic field near to the N pole of the magnet setting up the field, would be repelled by that pole and attracted by the S pole, consequently it would be carried along a line of force in the direction of the arrows shown in Fig. 23 B. As regards the intensity of the field, a very convenient quantitative meaning has been attached to the conception of lines of force which enables us to readily specify the magnitude as well as the direction of the field. Thus the intensity of a magnetic field is represented by the *number of lines of force* which are assumed to cut unit area—1 square centimetre—in a plane at right angles to the directions of the lines.

There are several methods of experimentally determining the intensity at any point of a magnetic field; thus, if a small, freely suspended compass needle (termed an exploring needle) be brought into any part of a field, it will be found that the needle vibrates rapidly, and it may be shown that the rate of oscillation is a measure of the intensity of the field at that point. Theoretically, however, the intensity of a magnetic field at any point is measured by the mechanical force in dynes which a unit magnetic pole would experience if placed at that point. Thus a field of unit intensity is such that it exerts a force of 1 dyne on a unit magnetic pole placed in it, and generally the strength or intensity of a field is denoted by the letter H; i. e. a field of H units strength is one that would exert a force of H dynes upon a unit magnetic pole placed in it. And, according to the convention already mentioned, each square centimetre of a field of H units strength would be cut (normally) by H lines of force. Unit field is thus represented by one line per square centimetre, and, generally, there are as many lines per square centimetre as there are dynes of force acting on unit pole at that particular point of the field.

Again, the force in dynes exerted on a magnetic pole of m units strength, when placed in a magnetic field where the strength is H units, is equal to the strength of the field H at that point in C.G.S. units multiplied by the strength of the pole, since the force exerted will be m times as great as the force exerted on a unit pole; that is,

$$F = \mathsf{H} \times m$$

and $$\mathsf{H}\text{ (strength of field at a point)} = \frac{\text{force in dynes}}{\text{strength of pole}}.$$

From this statement it is clear that if the strength of the pole be unity, then the same number denotes the strength of the field at a certain point, and also the force exerted at that point. This gives us a means of determining the strength of a field due to a magnet

(so long that only one pole need be considered) at any point in its field, since if we assume that a N pole of unit strength be placed at that point, and that the pole strength of the magnet be m, we have, then, by Coulomb's law,

$$f = \frac{m \times 1}{d^2},$$

but $$f = \mathsf{H}, \quad \therefore \mathsf{H} = m/d^2.$$

It thus follows that the strength of the field H, at a distance of 1 centimetre from a unit pole, is unity. We are now able to determine the number of lines of force which issue from a pole whose strength is m C.G.S. units. As we have seen, the value of H at all points distant d centimetres from the pole is

$$\mathsf{H} = m/d^2,$$

so that if we assume a sphere of radius r to be described about the pole as centre, a spherical surface of $4\pi r^2$ square centimetres will be pierced by H lines of force per square centimetre, and the whole spherical surface will therefore be pierced by

$$\mathsf{H} \times 4\pi r^2, \quad \text{or} \quad \frac{m}{r^2} \times 4\pi r^2;$$

that is, $4\pi m$ lines of force.

Therefore, the total number of lines of force emanating from a pole of m units strength is $4\pi m$. From a unit pole 4π lines of force leave or enter it. For purposes of comparison it is sometimes useful to determine the pole strength of a magnet per unit of polar area, which we shall now consider.

Intensity of Magnetization. This quantity may be briefly defined as *the magnetic moment of a magnet per unit volume*, and it has a determinate value depending upon the degree to which the magnet is magnetized. This definition introduces the term *magnetic moment*, which is defined as follows :—

The moment of a magnet is the product of its pole strength and the distance between the poles, and is the measure of the tendency of the magnet to rotate, when placed at right angles to the direction of a field of unit strength. That is, the magnetic moment of a magnet is

$$\mathrm{M} = m \times l.$$

The usual symbol for intensity of magnetization is I.

$$\therefore \mathsf{I} = \text{magnetic moment per unit volume}$$

$$= \frac{m\,l}{\mathrm{A}\,l} = \frac{m}{\mathrm{A}} = \text{pole strength per unit area of polar surface,}$$

where A = area of cross-section of the magnet assumed to be

constant, and through which the magnetization is supposed to be uniform,

$$\therefore m = \mathsf{I}\,\mathrm{A}$$

and the lines of force emanating from the magnet $= 4\,\pi\,\mathsf{I}\,\mathrm{A}$, and the lines of force per unit area of polar surface $= 4\,\pi\,\mathsf{I}$.

When the bar-magnet is of ordinary length and comparatively short, then both the poles exert some influence on the intensity of the field at different points, and since the poles are opposite in character the force exerted upon a unit-pole, and therefore the strength of the field at any point, is given by the resultant of two forces which have different directions. These directions have to be taken into account when making the determination of the field strength; the simplest case occurs for points along the axis produced, and if the length of the magnet be l cms. and the pole strength m, then at a point on the axis produced d cms., from the nearer pole the intensity of the field is

$$\mathrm{H} = \frac{m}{d^2} - \frac{m}{(d+l)^2}$$

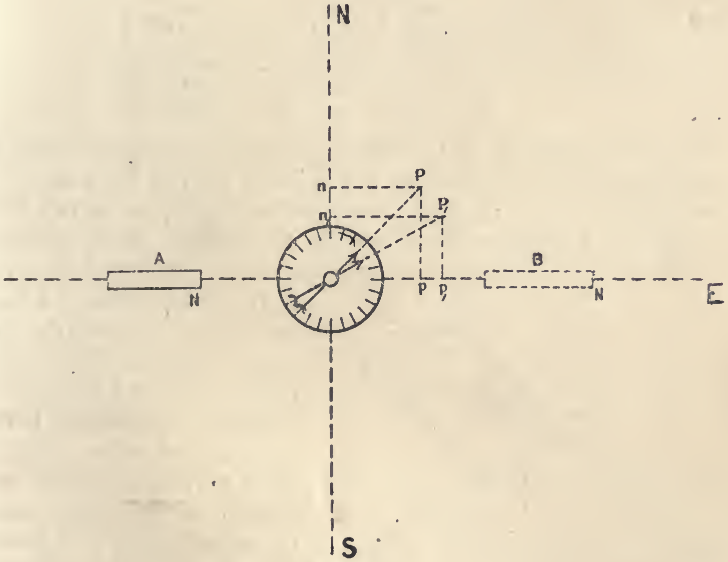


Fig. 24.

If the distance of the point from the centre of the magnet to the nearer pole be x cms., then

$$\mathrm{H} = \frac{m}{\left(x - \frac{l}{2}\right)^2} - \frac{m}{\left(x + \frac{l}{2}\right)^2} = \frac{2\,mlx}{\left(x^2 - \frac{l^2}{4}\right)^2}$$

To determine the intensity of a field, other than along the axis produced, the ordinary laws of mechanics for finding the resultant of two forces have to be applied. Thus, taking the case shown in Fig. 25, the two forces acting on a unit-pole at the point P make an angle θ with one another. For instance, the force of repulsion is $\frac{m}{d^2}$ in the direction along Pn; let the length Pn represent $\frac{m}{d^2}$ to a certain scale. Similarly the force of attraction is $\frac{m}{d_1^{\,2}}$ in the direction PS; let PS represent $\frac{m}{d_1^{\,2}}$. Then by the principles of the parallelogram of forces we obtain, by completing the parallelogram Pnos, the diagonal Po which represents in direction and magnitude the resultant of Pn and Ps. The magnitude of Po may be obtained by measurement, as is usual in graphical methods, or by calculation by means of the well-known formula

$$(\mathrm{P}o)^2 = (\mathrm{P}n)^2 + (\mathrm{P}s)^2 + 2\,\mathrm{P}n \times \mathrm{P}s\cos\theta$$

thus
$$(\mathrm{P}o)^2 = \left(\frac{m}{d^2}\right)^2 + \left(\frac{m}{d_1^{\,2}}\right)^2 + 2\frac{m}{d^2}\cdot\frac{m}{d_1^{\,2}}\cos\theta$$

and
$$\mathrm{P}o = m\sqrt{\frac{1}{d^4} + \frac{1}{d_1^{\,4}} + \frac{2\cos\theta}{d^2 d_1^{\,2}}}$$

So far the influence of the earth's magnetism has been ignored; when, however, the method of deflection is used to determine the intensity of a field or the pole strength of a magnet, the earth's force in a horizontal direction enters into the calculation. In this country the value of the horizontal component is usually taken as 0·18 unit.

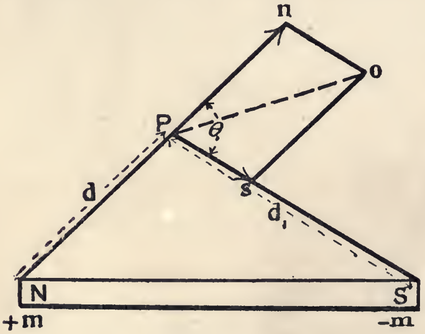


Fig. 25.

It is obvious that the magnetic field at a point due to a magnet may *coincide and augment, oppose and diminish or neutralize* the field due to the earth at that point. Experiment proves that a freely suspended needle always takes up a position of rest which coincides with the direction of the field in which it is placed; consequently if a bar magnet be placed with its axis in the E and W line, i. e. normal to the magnet meridian, there will be at all points along the axis produced a superposition of fields. Thus at O (Fig. 24) there is (1) the field due to the

earth's magnetism, the direction of which is N and S, and which is of constant intensity denoted by H, and also (2) a field due to the magnet A in the direction OE, the intensity of which depends upon the distance from O. Let F denote the strength of the field at O due to the magnet at A, then if a freely suspended magnetic needle be placed at O it will take up a position of rest coinciding with the direction of the resultant of the two fields, in other words, it will be deflected from the magnetic meridian; and let θ be the angle made by the needle with the meridian, i. e. nOP in the figure. From the diagram we have

$$\frac{F}{H} = \tan\theta \quad \text{and} \quad F = H\tan\theta$$

but
$$F = \frac{m}{d^2} - \frac{m}{(d+l)^2}$$

and
$$m = \frac{d^2(d+l)^2}{2\,dl + l^2}\,H\tan\theta$$

If $\theta = 45°$ then $F = H = \cdot18$, since $\tan 45° = 1$. If a second magnet be introduced at B, then the field at O will be increased or decreased according as its N pole points in the same direction as that of the first magnet or not. With such arrangements the *tangent law* holds; it is as follows:—

If two magnetic fields, the directions of which are at right angles to each other, are superimposed at a point, and a compass needle be placed at this point, it takes up a position which indicates the resultant field at that point, which is such that the axis of the needle makes an angle θ with the direction of one of the fields, the tangent of which is equal to the ratio of the other field to this field.

If the bar magnet be placed in the magnetic meridian with its N pole pointing to the N, then the field at a point in the meridian is given by F + H, whilst if the S pole of the magnet points to the N, then the intensity of the field along the meridian is given by F − H or H − F.

Worked Examples. (1) The force of attraction between two poles A and B is balanced by the weight of 0·3 grammes when placed 3 cms. apart, whilst the attraction between the pole A and a third pole C is equal to the weight of 0·25 grammes when placed 5 cms. apart. Determine the strengths of the poles B and C, if that of A is 109 units.

Since
$$F = \frac{m_a m_b}{d^2}$$

$$0{\cdot}3 \times 981 = \frac{109 \times m_b}{3^2}$$

and
$$m_b = \frac{0{\cdot}3 \times 981 \times 9}{109}$$
$$= 24{\cdot}3 \text{ units}$$

Again $$0{\cdot}25 \times 981 = \frac{109 \times m_c}{5^2}$$

and $$m_c = \frac{0{\cdot}25 \times 981 \times 25}{109}$$
$$= 56{\cdot}25 \text{ units}$$

(2) Determine the distance which the poles B and C (example 1) must be placed apart so that the force exerted between them may be one dyne.

$$1 = \frac{24{\cdot}3 \times 56{\cdot}25}{d^2}$$

and $$d = \sqrt{24{\cdot}3 \times 56{\cdot}25}$$
$$= 36{\cdot}97 \text{ cms.}$$

(3) Two equal magnetic poles of opposite kind are placed half a metre apart, and one pole is sufficient to set up a magnetic field of unit intensity at a point one metre away. Find the force of attraction in dynes.

Since $$\text{H} = \frac{m \times 1}{(100)^2} = 1$$
$$m = 10000 \text{ units}$$
$$\therefore \quad \text{F} = \frac{10000 \times 10000}{(50)^2}$$
$$= 40000 \text{ dynes.}$$

(4) Two N poles of strengths 90 and 40 units respectively are placed 10 cms. apart, determine the position of the point between them which is of zero intensity.

Let the point be x cms. from the stronger pole, then

$$\frac{90}{x^2} = \frac{40}{(10-x)^2}$$

and $$\frac{(10-x)^2}{x^2} = \frac{4}{9}$$

or $$\frac{10-x}{x} = \frac{2}{3}$$

and $$30 - 3x = 2x$$

whence $$x = 6 \text{ cms.}$$

(5) A bar magnet 8 cms. long is placed in the magnetic meridian with its N pole pointing N. At a point in the meridian 5 cms. distant from the N pole the intensity of the field is unity; determine the strength of the pole of the magnet.

Since the intensity of the field = H + F = 1

and $$\text{H} = 0{\cdot}18 \qquad \therefore \qquad \text{F} = 0{\cdot}82$$

$$\therefore \quad 0{\cdot}82 = \frac{m}{5^2} - \frac{m}{(5+8)^2}$$
$$= m\left\{\frac{1}{25} - \frac{1}{169}\right\} = \frac{144}{25 \times 169} m$$

and $$m = \frac{0{\cdot}82 \times 25 \times 169}{144}$$
$$= 24 \text{ units.}$$

EXERCISES IV A.

Magnetic Fields.

(1) A long magnetized needle is suspended from one end of the beam of a very delicate balance with its S pole pointing downwards, and 4·5 grammes are required to balance the needle. A piece of glass 1 cm. thick is placed horizontally so as to just touch the needle, and the N pole of another long needle is placed immediately under the first needle so as to just touch the underside of the glass, and 4·65 grammes are now required to just overcome the attraction between the two poles. Determine the force of attraction in dynes.

(2) A third needle replaces the second needle in the above case and 4·75 grammes are required to just overcome the attraction; determine the ratio of the attractions between the first and second, and between the first and third. Also compare the pole strengths of the second and third.

(3) With what force does a N pole whose strength is 15 units attract a S pole of 10 units strength, placed 5 cms. apart?

(4) A N pole of 20 units strength repels another pole placed 10 cms. away from it, with a force of three dynes. Find the strength of the second pole.

(5) The force of attraction between a N pole of a long magnet and an adjacent S pole of another long magnet 6 cms. apart was 8 dynes. What is the strength of each pole if the S pole was half the strength of the N pole?

(6) The force exerted between a S pole of 45 units strength, and a N pole 120 units strength is 24 dynes. Find the distance between them.

(7) A S pole of 60 units strength repels a similar pole 4 cms. from it with a force equal to the weight of a gramme. Find the strength of the second pole.

(8) If the distance between them be increased to 12 cms. calculate the force of repulsion between them.

(9) Two equal and opposite magnetic poles attract one another with a force of 16 dynes when 2·5 cms. apart. Find the strength of each pole.

(10) The force exerted between a S pole of 45 units strength and a N pole 245 units strength is equal to the weight of 50 milligrammes. Find the distance between them.

(11) A force equal to the weight of 4 ozs. is required to pull a ball of soft iron from contact with one of the poles of a magnet (A), and a force equal to the weight of 9 ozs. is required to pull the same ball

off one of the poles of a second magnet (B). Show what are the relative strengths of the poles A and B. (S. and A. 84.)

(12) Two magnetic poles are placed one centimetre apart and the force exerted between them is 15 dynes. How must the distance be varied so that the force between them may be 1·5 dynes?

(13) Two long magnets are placed vertically with their N poles (A and B) at the same level as the N pole (C) of a compass-needle, one being magnetic east and the other magnetic west of C. If the compass-needle is not deflected when the distance AC is twice BC, and if all the magnets are so long that the effects of their S poles may be neglected, show what are the relative strengths of the poles A and B. (S. and A. 87.)

(14) Two straight electromagnets are placed in a line with their N poles near each other, and at such a distance that they repel with a force equal to the weight of 10 lbs. Show what will be the force between the magnets at the same distance if the current circulating round each is increased sufficiently to magnetize it three times as strongly as before. (S. and A.)

(15) Find the intensity at a point in the magnetic field due to a certain magnet of 25 units pole strength 4 cms. from the N pole, the length of the magnet being such that the influence of the S pole may be neglected.

(16) Two short bar magnets the moments of which are 108 and 192 respectively are placed along two lines drawn on the table at right angles to each other. Find the intensity of the magnetic field due to the two magnets at the point of intersection of the lines, the centres of the magnets being respectively 30 and 40 cms. from this point. (S. and A.)

(17) What is the strength of a magnetic pole which experiences a force of 0·9 dynes when placed in the earth's magnetic field, where the intensity is 0·18?

(18) Find the strength of the field due to a bar magnet, 10 cms. long, and of 10 units strength, at a point 2 cms. from the N pole on the axis of the magnet produced.

(19) What is the strength of the field due to the above magnet, at a point 5 cms. distant from the centre of the magnet, on the normal line?

(20) What is the intensity at a point in the magnetic field due to a certain magnet of 25 units pole strength, 3 cms. from the N pole and 4 cms. from the S pole, the length of the magnet being 5 cms.?

(21) A magnet one decimetre in length has pole strengths of 5 units. Determine the intensity of the field at a point equidistant from the poles and $5\sqrt{3}$ cms. from the centre of the axis.

(22) Find the intensity of a magnetic field at a point C, produced by a thin bar magnet of 10 units strength, the poles A and B of which are 13 cms. apart, if the angle ACB is a right angle, and the distance AC 12 cms.

(23) A bar magnet is placed horizontally in the magnetic meridian, at a place where the earth's field (H) is 0·2, with its N pole pointing to the north. Find the intensity of the field at a point in the meridian 3 cms. from the N pole, if the magnet is 5 cms. long and of $11\frac{1}{4}$ units strength.

(24) A short bar magnet is placed on a table with its axis perpendicular to the magnetic meridian, and passing through the centre of a compass needle. In London the compass needle is deflected through a certain angle when the centre of the magnet is 25 inches from the centre of the needle. If the experiment be repeated in Bombay the magnet must be moved 5 inches nearer to the needle to produce the same deflection. Use these data to compare the horizontal forces in London and Bombay. (S. and A.)

(25) Two bar magnets of 20 and 25 units strength respectively are placed E and W with their N poles adjacent and 6 cms. apart. Find the position of zero field in the line joining their N poles, the lengths of the magnets being such that the influence of the S poles may be neglected.

(26) ABCD is a square of 4 cms. side. At A and C are placed N poles of 10 units strength, and at B a S pole of 8 units strength. What is the force acting on the pole at B?

(27) A magnet 12 cms. long is placed at right angles to the meridian at a place where $H = 0{\cdot}18$, and a small needle is deflected through 45° when the nearer pole of the magnet was 13·25 cms. from the centre of the needle. What was the pole strength of the magnet?

(28) A bar magnet 3 cms. long is placed E and W on one side of a small compass needle, at a place where H due to the earth's field is 0·21 units, so that when the nearer pole of the magnet is 2 cms. from the centre of the needle, the needle is deflected through an angle of 45°. What is the strength of the pole of the magnet?

(29) A magnet placed due east (magnetic) of a compass needle deflects the needle through 60° from the meridian. If at another station, where the horizontal force of the earth's magnetism is three times as great as at the first, the same magnet be similarly placed with respect to the compass needle, what will be the deflection of the latter? (S. and A.)

(30) If a sphere of 10 cms. radius is described about a magnetic pole of strength 100 units as centre, determine—(i) the mechanical attraction in dynes exerted upon a unit pole of opposite sign placed

at any point on the spherical surface; (ii) how many lines of force pass through each square centimetre of surface on the surface? (iii) what is the total number of lines of force emanating from the pole at the centre? (iv) if the pole at the centre were a unit pole, what would be the total number of lines of force? (v) if 4π lines of force pass through each square centimetre at the surface of the sphere, what would be the strength of the pole at the centre?

Section II. Magnetic Moment.

§ 37. **The Magnetic Moment of a Magnet.** Experiment shows that a magnetic needle, suspended at its centre by a torsionless fibre and free to oscillate in a horizontal plane, takes up a position of rest so that its axis coincides with the direction of the lines of force of the field in which it is placed. This is because a force of H m dynes acts on each pole in the direction of the lines of force of the field. Since the horizontal component of the earth field is constant the earth's field is said to be uniform, consequently it may be represented graphically by parallel straight lines.

The earth's action on a freely suspended needle is very clearly

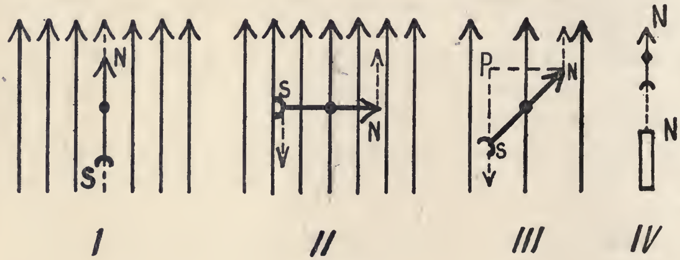


Fig. 26.

shown when the needle is placed so that its axis is at right angles to the magnetic meridian, for then a force of H m dynes in the opposite direction to that exerted by the field must be applied at both ends if we wish to keep it in that position. In other words, it is found that a *couple* (two equal, parallel, but opposite forces) tends to restore the needle to its position of equilibrium in the meridian. Now, every-day experience teaches that a force acting at right angles to a lever tends to rotate the lever, and in mechanics the tendency of a force to produce rotation about a point is termed the *moment of the force*, and numerically the moment of a force is equal to the product of the force into the perpendicular distance between the direction of the force and the point. Thus, in the case shown in Fig. 26 II, the magnetic

field sets up a tendency to rotate the needle, and the moment of the force acting at both ends is

$$\text{Moment of the force} = F \times \text{perpendicular distance}$$
$$= Hm \times \frac{l}{2} = \tfrac{1}{2}Hml \text{ dyne-centimetres.}$$

The effect of the two moments is to turn the needle in the same direction, so that the total tendency to produce rotation is therefore twice $\frac{1}{2}Hml$ or Hml dyne-centimetres. When two equal, parallel, but opposite forces, or a couple act on a magnetic needle, the torque is measured by the product of one force (Hm) into l (the arm of the couple). If the field be of unit strength, i. e. if $H = 1$, then the torque or moment of the couple is $m.l.$ or M. M or ml is termed the *moment of the magnet*, hence the definition :—

The moment of a magnet is numerically equal to the product of its pole strength and the distance between the poles, and is the measure of the tendency of the magnet to rotate, when placed at right angles to the direction of the lines of force in a field of unit strength.

The moment of the couple may be stated in general terms as follows. Let the axis of the needle make an angle a with the direction of the lines of force of the field, as in Fig. 26 III, then

$$\text{the moment of the couple} = Hm \times PN$$

but $$\frac{PN}{NS} = \sin a$$

and $$PN = NS \times \sin a = l \sin a$$

since NPS is a right-angled triangle, and

$$\text{the moment of the couple} = Hm \times l \sin a$$
$$= Hml \sin a = HM \sin a.$$

The maximum couple is therefore experienced when $a = 90°$, i. e. when the needle is at right angles to the direction of the lines of force.

§ 38. **Torsion Balance.** To determine the force exerted between two magnets it is customary to use a torsion balance, which is a kind of magnetic balance consisting of a magnetic needle suspended by a torsionless fibre with a conveniently placed scale for giving the angle of deflection, and also an index to indicate the angle of torsion given to the fibre. As is well known, wires or fibres when twisted exert a definite force to recover their original form, and the magnitude of this tendency depends upon the length and diameter of the fibre, the material of which the fibre is made and the angle through which the wire or fibre has been twisted. Advantage of this fact is taken

to displace a magnetic needle from its position of rest in the torsion balance, by giving a certain twist to the fibre, and from the relationship which exists between the torsional couple of the twisted fibre and the moment of the couple of the deflected magnet, numerous investigations respecting magnetic quantities may be made.

The moment of the couple of a twisted fibre, or torsional moment as it is called, is equal to $\tau\theta$, where θ is the angle of torsion, and τ is the force which would have to be applied at right angles to the end of a lever of unit length, suspended by the fibre, so as to balance the couple due to the torsional force in the fibre for 1°. In other words,

$$\tau\theta = \text{moment of the force of restitution,}$$

τ is obviously a constant for the fibre, and depends upon the nature of the material. It may be called the coefficient of torsion, and its magnitude is inversely proportional to its length and directly proportional to the fourth power of the diameter. If the suspended needle is l cms. long and a force f applied normally at one end of the end produces a deflection of θ degrees, the angle of torsion is θ, and

$$f \times \frac{l}{2} = \tau\theta$$

from which

$$f = \frac{2\,\tau\theta}{l} = \mathrm{k}\theta$$

In words, the force applied is proportional to the angle of torsion, and k or $\frac{2\tau}{l}$ is obviously a constant of the instrument known as the *reduction factor* of the instrument.

When a bar magnet is suspended by the fibre of a torsion balance, so that there is no twist when the magnet is in the meridian, let us suppose that the fibre must be twisted through A° at the top to produce a deflection of the magnet of $\theta°$, then

the angle of torsion for $\theta°$ deflection is $(\mathrm{A}-\theta)$ degrees

$\therefore$ " " " " " 1° " " $\frac{\mathrm{A}-\theta}{\theta}$ "

This quantity $\frac{\mathrm{A}-\theta}{\theta}$ is termed the earth's directive action, and $\mathrm{N} \times \frac{\mathrm{A}-\theta}{\theta}$ is the earth's directive action for N°.

If the magnet is displaced from the meridian by the repulsive force of another magnet the *torsional equivalent* of the repelling force is $\mathrm{N} + \frac{\mathrm{A}-\theta}{\theta} \times \mathrm{N}$.

When the magnet is deflected from the meridian by twisting the

suspending fibre at the top, the magnet takes up another position of rest when the moment of the couple of the magnet deflected is equal to the torsional moment of the twisted wire, that is

$$\mathrm{H}ml \sin a = \tau (\mathrm{A} - a)$$

Similarly, if by twisting the fibre through $\mathrm{A}_1°$ at the top the angle of deflection is a_1, then

$$\mathrm{H}ml \sin a_1 = \tau (\mathrm{A}_1 - a_1)$$

By combining the two results we get

$$\frac{\mathrm{H}ml \sin a}{\mathrm{H}ml \sin a_1} = \frac{\tau (\mathrm{A} - a)}{\tau (\mathrm{A}_1 - a_1)} \text{ or } \frac{\sin a}{\sin a_1} = \frac{\mathrm{A} - a}{\mathrm{A} - a_1}$$

Worked Examples. (1) Determine the magnetic moment of a bar magnet which requires a force of 25 dynes applied in a direction at right angles to the axis and at one end to displace it through 30° in the earth's field if the magnet is 10 cms. in length.

Here the moment of a force balances the moment of the couple exerted by the earth's magnetism upon the magnet, and

$$\begin{aligned} \text{moment of the force} &= \text{moment of couple} \\ \text{or } 25 \times 5 &= \mathrm{H}ml \sin a \\ &= \cdot 18\, ml \sin 30° \\ &= \cdot 18\, ml \times \frac{1}{2} \\ \therefore \quad ml &= \frac{125}{\cdot 09} = 1388 \cdot 8 \text{ units.} \end{aligned}$$

(2) Two magnets, whose moments are as 4 to 5, are fastened together at the centres so that their axes make an angle of 90° with each other. Determine the position taken up by the combination, if free to move in a horizontal plane, when under the action of the earth's magnetism. Also determine the moment of the system.

It is obvious that neither of the magnets can take up a position of rest with its axis in the meridian, and also that the moment of the couple of one magnet will be equal to that of the other when the system is at rest. Let the magnet of moment 4 make an angle a with the meridian, then the axis of the other will make an angle $(90° - a)$ with the meridian, and for equilibrium

$$\begin{aligned} \mathrm{H}m_1 l_1 \sin a &= \mathrm{H}m_2 l_2 \sin (90° - a) \\ \therefore \quad 4 \sin a &= 5 \sin (90° - a) \\ &= 5 \cos a \end{aligned}$$

and
$$\frac{\sin a}{\cos a} = \tan a = \frac{5}{4} = 1 \cdot 25$$

and from a table of tangents (p. 370) the angle whose tangent is 1·25 is approximately 51° 21′ 22″, consequently the combination of magnets take up a position of rest such that the magnetic meridian divides the right angle formed by the two magnets into two parts, i.e. 51° 21′ 22″ and 38° 38′ 38″.

A system of magnets has a magnetic moment which is equivalent to the moment of a single magnet having a definite axis, and any combination of

magnets will be acted upon as if the combination were replaced by its equivalent magnet. To determine the magnetic moment and the position of its equivalent axis of a system of magnets, the method adopted for finding the resultant of two or more forces in Mechanics must be applied. Thus, in Fig. 27, if OA and OB represent the moments of the two magnets, and the angle AOB be the angle formed by them, then OR represents the equivalent moment and also fixes the position of its axis relatively to the axes of the two magnets. Let the angles AOR and BOR be α and β respectively, then

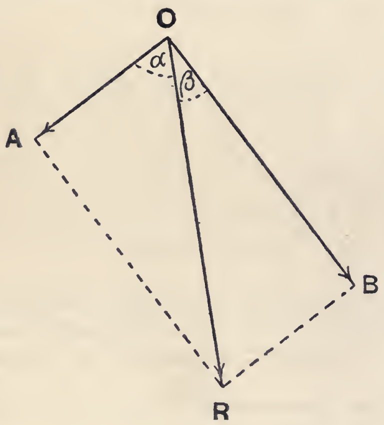


FIG. 27.

$$OA \times \sin \alpha = OB \times \sin \beta$$

or $$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha}$$

and $$(OR)^2 = (OA)^2 + (OB)^2 + 2 \times OA \times OB \times \cos (\alpha + \beta)$$

In the example given

$$4 \sin \alpha = 5 \sin \beta$$
$$= 5 \sin (90° - \alpha) = 5 \cos \alpha$$

and $$\tan \alpha = \frac{5}{4} = 1{\cdot}25$$

and $$\alpha = 51° \; 21' \; 22'' \text{ as before.}$$

Also $$OR = \sqrt{(OA)^2 + (OB)^2 + 2 \times OA \times OB \cos (\alpha + \beta)}$$
$$= \sqrt{4^2 + 5^2 + 2 \times 4 \times 5 \cos 90°}$$
$$= \sqrt{41} = 6{\cdot}4 \text{ units.}$$

(3) Two magnets are 12 cms. long and of the same strength. When one is placed as the suspended needle of a torsion balance, it is found that a couple of 144 dyne-centimetres is required to give to the wire a twist of 360°. Determine (i) the constant or reduction factor of the balance; (ii) the force which would give a twist of 60°; and (iii) the pole strength of the magnets if when the second magnet is placed vertically so that the two north poles are 1 cm. apart the deflection is 270°.

(i) Let f = the force applied at each end to produce a couple of 144 dyne-centimetres,

then $$f \times 12 = 144 \text{ and } f = 12 \text{ dynes}$$

but $$f = k\theta$$

$$\therefore \quad 12 = k \times 360 \text{ and } k = \frac{12}{360} = \frac{1}{30}$$

(ii) if the deflection = 60° the force required is

$$f_1 = k\theta_1 = \frac{1}{30} \times 60$$
$$= 2 \text{ dynes.}$$

(iii) Let f_2 = the force of repulsion when the two equal poles are 1 cm. apart

and m = the pole strength of the magnets

then $m^2 = f_2$ and $m = \sqrt{f_2}$

but
$$f_2 = k\theta_3 = \frac{1}{30} \times 270$$
$$= 9$$
$$\therefore \quad m = \sqrt{9} = 3 \text{ units.}$$

(4) The magnet in a torsion balance is deflected through 45° when the torsion head is turned through 180°, how much more must the torsion head be turned to deflect the magnet through 60°?

Let A_1 = angle through which torsion head must be turned for deflection of 60°, then
$$Hml \sin 60° = \tau (A_1 - 60°)$$
but
$$Hml \sin 45° = \tau (180° - 45°)$$
$$\therefore \quad \frac{\sin 60°}{\sin 45°} = \frac{A_1 - 60°}{135°}$$
but $\sin 60° = 0{\cdot}866$ and $\sin 45° = 0{\cdot}7071$ (p. 368).
$$\frac{0{\cdot}866}{0{\cdot}7071} = \frac{A_1 - 60}{135}$$
$$\therefore \; A_1 = \frac{0{\cdot}866}{0{\cdot}7071} \times 135 + 60 = 225°{\cdot}3$$
$\therefore$ torsion head must be turned through $225°{\cdot}3 - 180° = 45°{\cdot}3$.

(5) In one of Coulomb's experiments the torsion head had to be turned through 36° to give a deflection of 1°. When a magnet was introduced so as to deflect the needle the deflection was 24° and it was found that the deflection could be reduced to 12° by turning the torsion head through 8 complete revolutions. Show that the law of inverse square of the distance holds.

$$\text{Earth's directive force} = 36° - 1° = 35°$$
$$\left.\begin{array}{l}\text{Force balancing repulsion}\\ \quad\text{for } 24°\end{array}\right\} = \left\{\begin{array}{l}\text{torsion on wire + earth's}\\ \quad\text{directive action}\end{array}\right.$$
$$= 24° + 24(35°) = 864°$$
$$\left.\begin{array}{l}\text{Force balancing repulsion}\\ \quad\text{for } 12°\end{array}\right\} = \left\{\begin{array}{l}\text{torsion on wire + earth's}\\ \quad\text{directive action}\end{array}\right.$$
$$= (8 \times 360° + 12°) + 12\,(35°) = 3312°$$
and $3312 = 4$ times 864 approximately.

EXERCISES IV B.

(1) The moments of two magnets are as 3 : 1, and the lengths of the magnets are as 2 : 1. Compare the pole strengths.

(2) Two magnets have equal magnetic moments, but their lengths are as 3 : 4. Compare the pole strengths.

(3) A magnet 20 cms. in length and of 5 units pole strength is placed in the earth's field so as to make an angle of 30° with the

meridian. Determine the moment of the couple tending to restore it to the meridian.

(4) What would be the moment of the couple required to deflect the above needle at right angles to the magnetic meridian?

(5) A magnetic needle 4 cms. long and of 9 units strength is placed in a NE position. What force must be applied at right angles to one end of the needle to keep it there, if $H = \cdot 18$?

(6) The lengths of two magnets are as 2 : 3, and their pole strengths are as 5 : 3. Compare the moments of the couples tending to restore them to the meridian when deflected through 30°.

(7) Compare the couples tending to restore a needle 10 cms. long and of 8 units strength back to the meridian, (i) when deflected through an angle of 45°; (ii) and through an angle of 30° after the strength of the poles has been increased 41·4 per cent.

(8) A magnetic needle, free to move in a horizontal plane, is deflected from the meridian through angles of 30°, 45°, and 60° respectively, when placed at three different places on the earth's surface. If the moments of the couples at the three places are as 2 : 3 : 4, find the horizontal intensity of the earth's force at these places, given that H at the first place is 0·18 units.

(9) Find the magnetic moment of a bar magnet which experiences a couple of 25 dyne-centimetres when placed in a magnetic field of unit strength, so that its axis makes an angle of 30° with the direction of the field.

(10) Find the magnetic moment of a needle, 10 cms. long, which when placed in the earth's field so as to make an angle of 30° with the magnetic meridian, requires a force of 10 dynes applied at right angles to one end of the needle, to keep it in that position.

(11) A magnet free to move in a horizontal plane like a compass needle is turned from its position of equilibrium through 45°, the magnet being 10 cms. long, and the strength of the field 0·2 units. If the moment of the return couple is 70·7 dyne-centimetres, find the pole strength of the magnet.

(12) The N pole of a very long thin magnet of 100 units strength was placed so that its axis was at right angles to the magnetic meridian, and opposite to the centre of a magnetic needle free to move in a horizontal plane, and of 20 units strength. If the needle be 18 cms. long, and its centre 16 cms. distant from the N pole of the magnet, determine the moment of the couple tending to deflect the needle.

(13) A straight piece of watch-spring, 6 ins. long, is magnetized, and laid on a flat cork floating in water. The spring is now bent until its ends are 2 ins. from each other, and they are fixed at that

distance by a piece of thread ; the spring is then replaced on the cork. Compare the forces with which the spring tends to make the cork take a definite direction in each case. (S. and A. 82.)

(14) A uniformly magnetized bar of brittle steel is broken into two pieces, one twice as long as the other, and the pieces are fastened together at right angles to each other. How would the combination thus formed set itself under the action of the earth's magnetic force if made to float on water? (S. and A. 82.)

(15) Two equal bar magnets are fastened rigidly together at right angles. Show that the greatest couple exerted on them by the earth's horizontal force will be $\sqrt{2}$ times the greatest couple exerted on each separately.

(16) Two magnets, whose moments are as 3 to 4, are fastened together, with their axes making an angle of 120° with each other. Show how to determine (graphically or otherwise) the position of equilibrium of the combination when free to move in a horizontal plane.

(17) The lengths of two needles are as 2 to 3, and their pole-strengths as 3 to 4. The shorter needle is floated on cork in water, and the force with which it takes up its position of rest is found to be F. The two needles are now connected like an astatic combination, i. e. so as to lie parallel to one another, with their unlike poles adjacent, and their centres in the same vertical plane. Determine the force with which the combination takes up its position of rest when placed on cork in water, as in the first case.

(18) The two needles are then connected similarly, but with their like poles pointing in the same direction. Determine the force with which this combination takes up its position of rest when placed as in the previous case.

EXERCISES IV c.

(1) A magnet 8 cms. long is placed in a torsion balance, and it requires a couple of 120 dynes to twist the wire through 360° ; determine the constant or reduction-factor of the balance. Also calculate the force which would twist the wire through 60°.

(2) If, in a torsion balance, the upper end of the wire must be turned through 90° to deflect a needle suspended at the other end through 15°, what is the amount of torsion per degree which measures the earth's directive action?

(3) A certain N pole is placed 1 cm. from the N pole of the torsion magnet, and the wire has to be twisted through 540° to bring the magnet to its normal position. If the reduction-factor of the torsion

balance is ·025, determine the strength of the two poles which are equal.

(4) Two similar poles are of such strength that, when placed in a torsion balance, the needle is deflected through 30°. How much torsion must be put on the wire to bring the needle back to 20° and 15°, if the earth's directive action per degree is measured by 15°?

(5) A bar magnet suspended by a fine wire points N and S (magnetic) when the wire is not twisted. When the upper end of the wire is turned through 100° the magnet is deflected through 30° from the meridian. Show how much the upper end of the wire must be turned to deflect the magnet 90° from the meridian. (S. and A. 84.)

(6) Two straight pieces of steel, one 3 inches and the other 5 inches long, are cut from the same narrow strip of steel. After being equally magnetized they are hung horizontally one at a time by the same fine glass thread, so as to rest in the magnetic meridian when the glass thread is not twisted. On turning the upper end of the thread half round (through 180°) the shorter magnet is deflected 10° from the meridian. Show how much the upper end of the thread must be turned to deflect the longer magnet ten degrees. (S. and A. 81.)

(7) Two magnets, A and B, are in turn suspended horizontally by a vertical wire so as to hang in the magnetic meridian. To deflect the magnet A through 45°, the upper end of the wire has to be turned once round. To deflect B through the same angle it has to be turned round one and a half times. Compare the moments of the two magnets. (S. and A. 87.)

(8) A magnet, suspended by a fine vertical wire, hangs in the magnetic meridian when the wire is untwisted. If, on turning the upper end of the wire half round, the magnet is deflected through 30° from the meridian, show how much the upper end of the wire must be turned in order to deflect the magnet 45° and 60° respectively. (L. U. 84.)

(9) A bar magnet is suspended in a torsion balance by a wire without torsion. When the torsion-head is turned through 360°, the bar is deflected 30° from the meridian. Through how many degrees must the torsion-head be turned that the magnet may be in equilibrium at right angles to the meridian? (L. U. 87.)

(10) A bar-magnet is supported horizontally by a fine vertical wire. The magnet lies in the magnetic meridian when there is no torsion in the wire, and the top of the wire must be turned through 100° in order to deflect the magnet 15° from the meridian. The magnet is removed, remagnetized and replaced, and now the upper end of the

wire has to be twisted through 150° to produce the same deflection of the bar. Compare the moments of the bar magnet in the two cases. (S. and A. 91.)

(11) A bar magnet, hung horizontally by a fine wire, lies in the magnetic meridian when the wire is without twist. It is then found that when the top of the wire is twisted through 120° the magnet is deflected through 30°. Through what further angle must the top of the wire be twisted in order to turn the magnet perpendicular to the magnetic meridian? (S. and A. 93.)

(12) A bar magnet is suspended horizontally in the magnetic meridian by a wire without torsion. To deflect the bar 10° from the meridian the top of the wire has to be turned through 180°. The bar is removed, remagnetized, and restored, and the top of the wire has now to be turned through 250° to deflect the bar as much as before. Compare the magnetic moments of the bar before and after remagnetization. (S. and A. 94.)

(13) With a certain magnet a deflection of 5° was produced when the torsion head was moved through 80°. On remagnetizing the bar to three times its former strength, through what angle must the torsion head be turned to deflect the magnet 5°?

§ 39. The Oscillations of a Magnet in a Magnetic Field. When a magnetic needle is freely suspended at its centre in a magnetic field, it oscillates about its position of rest if displaced from the meridian for a short time, and when the arc of vibration is small the oscillations of the needle are regulated by the same laws as those of a pendulum oscillating under the influence of gravity. Consequently (1) the square of the complete number of oscillations is proportional to the force producing the oscillations, and (2) the oscillations of a vibrating needle are isochronous, i. e. the time for a complete oscillation is the same for different amplitudes of oscillation.

Now the force producing the oscillations depends upon the intensity of the field and upon the magnetic moment of the magnet, and it may be proved experimentally that the following relation holds for a vibrating magnet, of moment M oscillating in a field of H units strength.

$$HM = 4\pi^2 n^2 I$$

where n = number of complete oscillations per second, and I is the moment of Inertia of the magnet—a quantity depending upon the mass of a body and upon the distances of the particles forming the body from the axis of rotation.

If t is the time of one complete oscillation $n = \frac{1}{t}$ and

$$t = 2\pi \sqrt{\frac{I}{MH}}$$

$$\therefore MH = K \times \frac{1}{t^2} = Kn^2$$

If the same magnet be set vibrating in two different fields of strengths, H_1 and H_2, and the number of oscillations be n_1 and n_2 respectively, then

$$MH_1 = Kn_1^2 \quad \text{and} \quad MH_2 = Kn_2^2$$

and
$$\frac{H_1}{H_2} = \frac{n_1^2}{n_2^2} = \frac{t_2^2}{t_1^2}$$

If the second field strength H_2 be the resultant of two fields, say that of the earth and also an intensity due to a magnet, then $H_2 = H_1 + F$ or $H_1 - F$ or $F - H_1$, if the magnet be placed with its axis in the meridian. In such a case let N be the number of oscillations made in a given time by the needle oscillating under the action of the earth alone, and n_1 those due to the action of the resultant field; then when the directions of the two fields are coincident

and
$$\left.\begin{aligned} MH_1 &= KN^2 \\ M(H_1 + F) &= Kn_1^2 \end{aligned}\right\} \quad \text{and} \quad \frac{H_1 + F}{H_1} = \frac{n_1^2}{N^2}$$

and
$$\frac{F}{H_1} = \frac{n_1^2}{N^2} - 1 = \frac{n_1^2 - N^2}{N^2}$$

If the direction of the field due to the magnet is opposed to that of the earth and $F_1 > H_1$ then

and
$$\left.\begin{aligned} MH_1 &= KN^2 \\ M(F_1 - H_1) &= Kn_2^2 \end{aligned}\right\} \quad \text{and} \quad \frac{F_1 - H_1}{H_1} = \frac{n_2^2}{N^2}$$

and
$$\frac{F_1}{H_1} = \frac{n_2^2}{N^2} + 1 = \frac{n_2^2 + N^2}{N^2}$$

Worked Example. Two magnetic poles A and B are compared by the method of oscillations, the two poles being placed in turn a certain distance from the oscillating needle. When the pole A is used the number of oscillations is 30 per minute, and 22 per minute when the pole B is used. The needle makes 10 oscillations per minute under the action of the earth alone. Compare the pole strengths.

$$M(H_1 + F_a) = Kn_a^2$$
$$M(H_1 + F_b) = Kn_b^2$$
$$MH_1 = KN^2$$

$$\therefore \quad \frac{H_1 + F_a}{H_1} = \frac{n_a^2}{N^2} \text{ and } \frac{F_a}{H_1} = \frac{n_a^2 - N^2}{N^2}$$

also
$$\frac{H_1 + F_b}{H_1} = \frac{n_b^2}{N^2} \text{ and } \frac{F_b}{H_1} = \frac{n_b^2 - N^2}{N^2}$$

$$\frac{F_a}{F_b} = \frac{n_a^2 - N^2}{n_b^2 - N^2}$$

Substituting the values of 30, 22 and 10 for n_a, n_b and N we get

$$\frac{F_a}{F_b} = \frac{30^2 - 10^2}{22^2 - 10^2} = \frac{800}{384}$$

$$= 2{\cdot}08$$

but $$\frac{m_a}{m_b} = \frac{F_a}{F_b} \qquad \therefore \qquad m_a = 2{\cdot}08\, m^b$$

If $$H_1 = 0{\cdot}18$$

then $$F_a = \frac{n_a{}^2 - N^2}{N^2} H_1 = \frac{30^2 - 10^2}{10^2} \times 0{\cdot}18$$

$$= 8 \times {\cdot}18 = 1{\cdot}44 \text{ units}$$

and $$F_b = \frac{22^2 - 10^2}{10^2} \times 0{\cdot}18 = 3{\cdot}84 \times {\cdot}18$$

$$= 0{\cdot}6912 \text{ unit.}$$

EXERCISES IV D.

(1) A small magnet freely suspended at its centre of gravity makes 14 oscillations in 2 minutes at London, and at Glasgow 20 oscillations in 3 minutes. If the value of H at London is 0·18, find the value of H at Glasgow.

(2) At a place where H = 0·18, a small needle makes 90 oscillations in 6 minutes under the action of the earth alone. When one pole of a long bar magnet is brought near the needle it makes 45 oscillations in 2 minutes. Determine the strength of the field due to the magnet at this point, and show what proportion the force exerted on the needle by the magnet bears to that exerted by the earth.

(3) A needle makes 10 oscillations per second under the influence of the earth's magnetism alone. When a magnet A is introduced into the field the oscillations are increased to 40 per second. When A is replaced by another magnet B the needle oscillates 60 times per second. Compare the pole strengths of A and B.

(4) To compare the intensities of two fields, two magnets A and B are each suspended in turn at a point in the earth's magnetic field, A making 20 oscillations per minute, and B 15 per minute. A is then suspended in one of the fields and makes 10 oscillations per minute, whilst B suspended in the other field makes 10 oscillations per minute. Compare the intensities of the two fields.

(5) A needle makes 41 oscillations per minute when the N pole of a bar magnet is placed a certain distance away, and 24 when the position of the bar magnet is reversed. Find the time of an oscillation under the action of the earth's force alone.

(6) A needle under the earth's influence made 15 oscillations per minute. The S pole of a long magnet placed 4 inches from the

needle increased the oscillations to 41 per minute. When placed 8 inches away the oscillations were 24 per minute. Prove the law of inverse squares.

(7) A magnetic needle balanced horizontally at its centre upon a firm point makes 11 vibrations in 2 minutes 1 second at a place A, and 12 vibrations in 2 minutes at a place B. Compare the strength of the earth's horizontal magnetic force at the two places, explaining clearly how you arrive at your result. (S. and A. 79.)

(8) A bar magnet which can move only in a horizontal plane is caused to vibrate at three different stations, A, B, and C. At A it makes 20 vibrations in 1 minute 30 seconds; at B, 25 vibrations in 1 minute 40 seconds; at C, 20 vibrations in 2 minutes. Find three numbers proportional to the forces which act upon the magnet at the three places. (S. and A. 88.)

(9) A small magnetic needle makes 100 oscillations in 5 minutes 36 seconds under the influence of the earth's magnetism only, and 100 vibrations in 4 minutes 54 seconds when a horizontal bar magnet is placed with its centre vertically below the needle and with its axis in the magnetic meridian. Compare the magnetic force exerted upon the needle by the bar magnet when in this position with that exerted upon the needle by the earth.

(10) A mariner's compass is placed on a table, and a bar magnet is laid on the floor below it, the centre of the bar magnet being straight underneath the centre of the compass-needle. When the N end of the bar magnet is northwards the compass-needle after being disturbed makes 100 oscillations in 16 minutes. When the N end is southwards the compass makes 100 oscillations in 12 minutes. When the bar magnet is removed so that the needle is under the influence of the earth alone, how long will it take to make 100 oscillations? (S. and A. 83.)

(11) A magnetic needle makes a complete oscillation in a horizontal plane in 2·5 seconds under the influence of the earth's magnetism only, and when the pole of a long bar magnet is placed in the magnetic meridian in which the needle lies, and 20 centimetres from its centre, a complete vibration is made in 1·5 seconds. Assuming $H = \cdot 18$ C. G. S. units, and neglecting the torsion of the fibre by which the needle is suspended, determine the strength of the pole of the long magnet. (L. U. 86.)

(12) Compare the magnetic moments and intensities of magnetization of two magnets, respectively $10 \times 3 \times 0{\cdot}5$ and $20 \times 4 \times 0{\cdot}7$ cms., which, when pivoted so as to oscillate horizontally in the earth's field about the shortest axis, are found to have the same time of oscillation.

CHAPTER V

THE MAGNETIC CIRCUIT

Section I. Electromagnetism.

The fundamental conception in the modern treatment of the principles of applied magnetism is that the passage of magnetic force is of the nature of a flux—a flow of magnetic force—and in this sense a magnetic field is permeated with magnetic flux, and lines of magnetic force are simply magnetic flux paths. In the same way that there is a rate of fall of potential corresponding to the passage of electricity in an electric circuit—a current of electricity also being of the nature of a flux—so there is also a rate of fall of magnetic potential between any two points in a magnetic system ; in this and in other ways complete magnetic systems are analogous to electric circuits, and a closed magnetic system is now termed a magnetic circuit. Furthermore, as we shall presently show, the relationship existing between the three essential elements of a magnetic circuit is identical with the relationship existing between electric pressure, current and resistance, and known as Ohm's Law, in the electric circuit, and consequently it is now customary to speak of Ohm's Law of the magnetic circuit. This conception of the magnetic circuit, and of Ohm's law of the magnetic circuit, very much simplifies the methods of treating this branch of applied science quantitatively, and facilitates in a great measure electromagnetic calculations in dynamo design.

§ **40. Electricity and Magnetism.** Although it had been known for a very long time that a very close connexion existed between electricity and magnetism, it was not until after Oerstedt, of Copenhagen, in 1819, made the discovery that currents of electricity radiate electromagnetic wave systems, and set up and maintain magnetic fields around the conductors through which they are passing, that the true connexion between electricity and magnetism was understood. The character of the magnetic field set up by electricity passing through a very long straight conductor is shown in Fig. 28 *a* as given by iron filings, which, being arranged as concentric circles around the wire as centre, indicate circular lines of force, whilst in Fig. 28 *b* a plan of the same field is given, the current flowing down-

wards as indicated. In this figure the direction of the lines of force are indicated by the arrow-heads, and the deflecting influence upon a freely suspended needle is also shown. Changing the direction of the current changes the direction of the lines of force, and it should be noticed that if we combine the relative directions of the current and of the lines of force we obtain an identical corkscrew motion, from which we get a simple rule for determining the direction of a magnetic field produced by a straight current, known as the corkscrew rule, and introduced by Maxwell.

A current flowing in a conductor sets up magnetic force round the conductor, the direction of which is the same as the direction of rotation of a right-handed corkscrew moving in the same direction as the current.

The magnetic influence extends some distance from the wire, and the intensity at any point of the field depends upon the strength of the current and its distance from the wire. The equation connecting

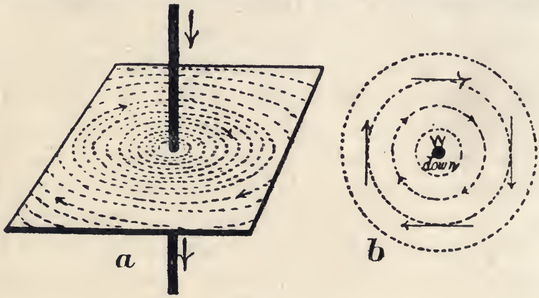


FIG. 28.

H with the current C^1 and the distance r may be proved mathematically and experimentally to be

$$H = \frac{C^1}{\frac{r}{2}} \text{ or } \frac{2\,C^1}{r}$$

if C^1 be the current in absolute units, and r the distance in centimetres. Since 1 ampere is equal to $\frac{1}{10}$ of a C.G.S. unit of current

$$H = \frac{2\,C}{10\,r}$$

if C be given in amperes.

If the conductor through which the current passes be looped into a single circular coil as in Fig. 29, so that each element of the circle is at the same distance from the centre, the strength of the field at the centre may be shown to be

$$H = \frac{C^1 l}{r^2} = \frac{C^1 \times 2\,\pi\,r}{r^2} = \frac{2\,\pi\,C^1}{r}$$

if C^1 be given in absolute units, and l is the mean circumference of the coil of r centimetres radius. If C be given in amperes

$$H = \frac{2\pi C}{10r}$$

The lines of force due to a circular current are shown in Fig. 29.

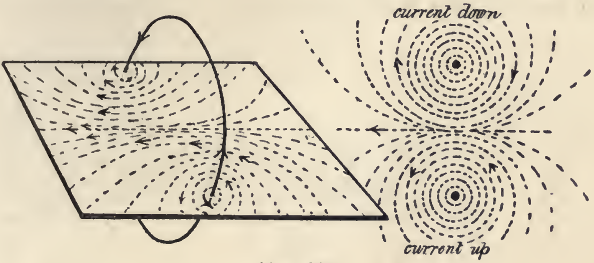


FIG. 29.

We may also find the intensity of the magnetic field due to a circular current at any other point on the axis of the coil. In

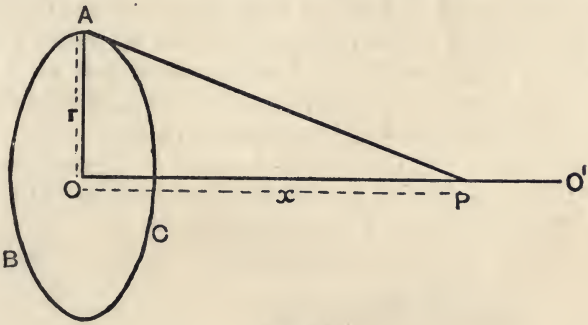


FIG. 30.

Fig. 30 let A B C be the coil of wire traversed by a current of C amperes, then the intensity of the field at a point P on the axis O O^1 may be shown to be

$$H = \frac{2\pi C r^2}{10(x^2 + r^2)^{\frac{3}{2}}}$$

where r = radius of the coil in centimetres, and x = the distance O P in centimetres. It will be observed that the distance A P = $(x^2 + r^2)^{\frac{1}{2}}$. When the wire is coiled as in Fig. 31, we have an electromagnetic spiral, helix or solenoid, and upon investigating the magnetic field which is set up when a current circulates through the coils, we find that within the coil it is intense and uniform, and

that its direction is parallel to the axis of the coil, except at the ends. The lines of force are shown by means of dotted lines in the figure. Externally the magnetic field exactly resembles that of an ordinary bar magnet, and so far as its magnetic properties are concerned, the coil is an ironless magnet so long as the current lasts. It is important to notice that the magnet field and its properties completely disappear immediately the current ceases. In practice the intense

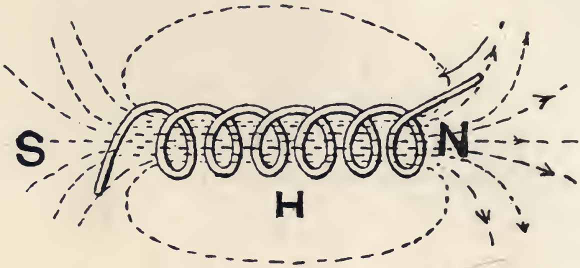


FIG. 31.

magnetic fields of dynamos and motors are produced by means of electromagnets, and it is obviously important to be able to determine readily the intensity of the magnetic field set up within a solenoid traversed by a current, both when iron is present and when absent. When the direction of the current is known, the polarity of a solenoid may be readily determined by the following rule:—

Grasp the solenoid with the right hand so that the fingers point in the same direction as the current, then the thumb outstretched at right angles points in the direction of the N pole of the solenoid.

Now, the lines of force threading a solenoid have a definite density

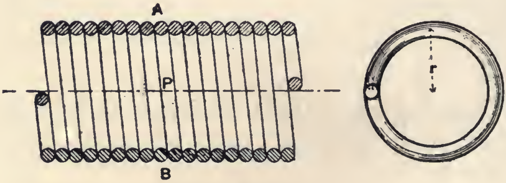


FIG. 32.

as well as a definite direction for a given current and number of turns of wire or spires, and the relationship between the intensity of the magnetic field at any point set up by the current traversing the coil (or, what is the same thing, the magnetic force exerted at that point), and the current and the number of turns may be determined as follows:—

Let Fig. 32 represent in longitudinal and cross section a straight

solenoid extending to infinity in both directions. Suppose the mean radius to be r centimetres, and that a constant current of C amperes traverses each turn. It is required to determine the intensity of the magnetic field at the point P on the axis. Obviously each current-carrying turn supplies a component of the intensity at P, the maximum component being, from what we have already said

$$h = \frac{2\pi C}{10\,r}$$

It is also clear that the magnitudes of the other components diminish as the perpendicular distances of the planes of the successive spires from P increases, and ultimately become very small. The component due to any spire or turn is given by

$$h_1 = \frac{2\pi C r^2}{10\,(x^2 + r^2)^{\frac{3}{2}}}$$

where x = distance of that turn from P.

To determine the total intensity at P, we have to find the sum of all the terms given by the last equation for all the turns supplying a share in the intensity at P, and this may be shown to be

$$\mathrm{H} = \frac{4\pi CS}{10\,l} \quad \text{or} \quad \frac{4\pi}{10}\,CS^1,$$

where S = total number of spires or turns,

l = the length of the solenoid in centimetres,

and S^1 = the number of turns per centimetre length.

As the formula is so important, we also give another method of obtaining the same result. Suppose we have a single turn of wire traversed by a current of C^1 absolute units, and enclosing an area of A square cms.; also let the thickness of the coil be t. Now it may be proved experimentally that, so far as external effects are concerned, this current-carrying coil may be replaced by a thin magnet, or magnetic shell of the same contour, and of strength C^1. Since the thickness of this shell may be chosen arbitrarily, let it be taken equal to t, then if the magnetic density of the faces of the shell be σ, we have

$$\sigma t = C^1$$

and

$$\sigma = \frac{C^1}{t}$$

Increasing the number of turns of wire traversed by the current to S spires or turns, we have:

$$\frac{1}{t} = S^1 \quad \text{turns per unit length}$$

and

$$\sigma = C^1 S^1$$

Corresponding to this solenoid we shall have an equivalent magnet composed of S magnetic shells, in which the adjacent faces of the neighbouring shells neutralize each other ; let the pole strength of this equivalent magnet be m, then

$$m = \sigma \mathrm{A} = \mathrm{C}^1 \mathrm{S}^1 \mathrm{A}$$

and since
$$\mathrm{S}^1 = \frac{\mathrm{S}}{l}$$

$$m = \frac{\mathrm{C}^1 \mathrm{S} \mathrm{A}}{l}$$

Now, the intensity of a magnetic field at a point d cms. from a magnetic pole is $\frac{m}{d^2}$, and since magnetic force is radiated in all directions from a magnetic pole it follows that the intensity at any point on the surface of an imaginary sphere of unit radius described about the pole as centre will be m units. But the surface of this sphere is 4π square cms., therefore $4\pi m$ lines of force emanate from a magnet of pole strength m. We have therefore

$$4\pi m \text{ or } \frac{4\pi \mathrm{C}^1 \mathrm{S} \mathrm{A}}{l}$$

as the total number of lines of force within the solenoid due to the current C^1 absolute units traversing it. Let this number be N^1, and obviously

$$\mathsf{H} = \frac{\mathrm{N}^1}{\mathrm{A}} = \text{intensity of field within the solenoid}$$

and
$$\mathsf{H} = \frac{\mathrm{N}^1}{\mathrm{A}} = \frac{4\pi \mathrm{C}^1 \mathrm{S}}{l}$$

$$= \frac{4\pi \mathrm{C} \mathrm{S}}{10\, l} = 1{\cdot}257 \frac{\mathrm{C} \mathrm{S}}{l}$$

= 1·257 (or approximately $1\frac{1}{4}$) times the number of ampere-turns per centimetre length of the solenoid if C be given in amperes.

This formula gives the number of lines of force per square centimetre within the spiral, but it is not correct, except at the centre, unless the length of the spiral is many times its diameter, and to avoid any demagnetizing effects of the ends, and to get a clear conception of the modern views respecting the magnetic fields produced by such a current-carrying coil, we shall imagine the solenoid to be completely closed, and to form a closed ring by being wound upon a ring of wood or other non-magnetic substance. We shall then have a simple magnetic circuit in which magnetic force and a magnetic field are set up by a spiral current.

§ 41. **Magnetic Circuit, Magnetic Flux, Magnetomotive-force.** The fundamental idea of the modern theory of a magnetic circuit is that the spiral current is the *agency, cause* or *force* that sets up the physical condition known as a magnetic field. In all cases the field has a definite direction and intensity, and the conception of representing a magnetic field as if that space were filled with lines of force has led to the introduction of the idea (borrowing from the subject of fluid motion) that the propagation of magnetic force is of the *nature of a flux—a flow of lines of force*—and in this sense lines of force are magnetic flux paths, and a magnetic field is permeated by magnetic flux. Thus magnetic flux is the *effect* of which the force exercised by the spiral current is the *cause*. This cause, which is analogous to electromotive force (E.M.F.) in the electric circuit setting up an electric flux or current, is known as *magnetomotive force* (M.M.F.), a term introduced by Bosanquet to denote the *total magnetizing influence*. Magnetic flux is always continuous and always forms complete circuits, and in the case of the ring the direction of the magnetic force coincides with the direction of the circular axis of the circular helix. Magnetic flux is, therefore, a circuital quantity, and the path traversed by magnetic flux is known as a *magnetic circuit*. Magnetic flux is measured in terms of a unit called a *Maxwell*, which will be defined later.

Now, since

$$\mathsf{H} = \frac{\frac{4\pi}{10}\,\mathrm{CS}}{l}$$

we see that the value of H, the magnetic force or number of lines of force set up by the spiral current, measures the magnetizing force or M.M.F. per centimetre length of the magnetic circuit ; H is therefore the *slope of the magnetomotive-force* (M.M.F.), and is sometimes termed the *magnetomotive-force* (M.M.F.) *intensity*.

$$\therefore \quad \mathsf{H} = \frac{\text{M.M.F.}}{l}$$

and M.M.F. $= \mathsf{H} \times l =$ magnetic force × length

$$= \frac{4\pi}{10}\,\mathrm{CS}$$

from which we see that the measure of the magnetomotive-force (M.M.F.) round the circuit is found by taking the product of the magnetic force (H) and the mean length of the magnetic circuit in centimetres. Magnetomotive-force, the analogue of electromotive-force, is the magnetizing power of the *excitation, or ampere-turns*, as the product of the current in amperes and the number of turns of wire forming the solenoid is called. Experimentally it may be

shown that the magnetizing force is constant so long as the number of ampere-turns is the same, whatever the individual values of the current and number of turns of wire may be. Thus 5 amperes through 100 turns of wire produce the same magnetomotive force as 4 amperes through 125 turns of wire. The C.G.S. unit of M.M.F. may be stated to be equal to 0·79577 or 0·8 of an ampere-turn.

$$\text{Since} \quad \frac{4\pi}{10} = 1{\cdot}257, \quad \text{and} \quad H = \frac{\frac{4\pi}{10}\,CS}{l} = \frac{1{\cdot}257\ CS}{l}$$

the intensity of the magnetic field or of the magnetic force at the centre of a long solenoid is equal to 1·257 times (or $1\frac{1}{4}$ times approximately) the number of ampere-turns per centimetre of the length of the solenoid. When the length of the solenoid is given in inches a very convenient relationship holds, which we may determine as follows

Let l' = length of the solenoid in inches,
then $2{\cdot}5\,l'$ = ,, ,, ,, ,, centimetres,

$$\text{and} \qquad H = \frac{\frac{4\pi}{10}\,CS}{2{\cdot}5\,l'} = \frac{1{\cdot}257\ CS}{2{\cdot}5\,l'} = \frac{1}{2}\frac{CS}{l'}\ \text{(approx.)}$$

= number of ampere-turns per $\frac{1}{2}$ inch of length.

H is here, of course, the number of lines of force per square centimetre, but if it be necessary to give the number of lines of force per square inch, the relationship is as follows :—

Since 1 sq. inch = $(2{\cdot}54\ \text{cms.})^2$ = 6·4516 sq. cms.

∴ if H_{11} = lines of force per sq. inch,

$$H_{11} = \frac{6{\cdot}4516 \times 1{\cdot}257\ CS}{2{\cdot}54\,l'}$$

$$= \frac{2{\cdot}54 \times 1{\cdot}257\ CS}{l'}$$

$$= 3{\cdot}192\frac{CS}{l'}\ \text{(lines per sq. inch).}$$

The C.G.S. unit of field strength is termed the *Gauss.*

Worked Examples. A current of 5 amperes is sent through a helix consisting of 250 spires or turns, and which is 50 cms. long. What is the intensity of the field at the centre of the coil, and what is the total flux if the area of the cross section of the coil is 3 sq. cms. ?

$$\text{Since} \quad \text{the intensity} = H = \frac{0{\cdot}4\,\pi\ CS}{l} = \frac{1{\cdot}257\ CS}{l}$$

$$= \frac{1{\cdot}257 \times 5 \times 250}{50}$$

= 31·425 lines per sq. centimetre.

And, since the area of cross-section = 4 sq. cms.
the total flux of lines $= \mathsf{HA} = N^1 = 4 \times 31{\cdot}425$
$= 125{\cdot}7$.

(2) Calculate the number of ampere-turns of excitation required to set up 50 lines per square centimetre inside a hollow helix, bent so as to form a complete ring 20 centimetres mean diameter.

Evidently the length of the magnetic circuit is given by the mean circumference of the ring formed by the coil, and since its mean diameter is 20 cms.

$$l \text{ (length of helix)} = 20\,\pi \text{ cms.}$$

and

$$\because \quad H = \frac{0{\cdot}4\,\pi\,CS}{l} = \frac{0{\cdot}4\,\pi}{l} \times CS$$

$$\therefore \quad CS = \frac{H \times l}{0{\cdot}4\,\pi} = \frac{50 \times 20\,\pi}{0{\cdot}4\,\pi}$$

$$= \frac{50 \times 20}{0{\cdot}4} = 2500 \text{ ampere-turns.}$$

EXERCISES V A.

(1) A magnetic pole of 10 units strength is placed in magnetic fields of 0·1, 10, and 1000 C.G.S. units strength respectively, determine the mechanical force in dynes and grammeweights experienced in each case.

(2) A spiral of uniformly wound wire of 1000 turns and 1 metre long is traversed by a current of 10 amperes. Determine

(*a*) the total M.M.F. exerted,

(*b*) the force exerted on unit magnetic pole placed at the centre,

(*c*) the total magnetic flux.

(3) What is the intensity of the magnetic field inside a helix of one ampere-turn per centimetre length?

(4) How many amperes will be required to produce 125 lines in a helix 1 metre long consisting of 1000 turns, if the cross-sectional area is 2·5 square centimetres?

(5) Prove that 80 ampere-turns per centimetre = 100 ampere-turns per $\frac{1}{2}$ inch.

(6) Prove that for long solenoids the internal magnetic flux-density for air is $\frac{CS}{0{\cdot}8l}$ maxwells per sq. cm.

(7) Determine the flux-density in maxwells per sq. cm. when the excitation is 80 ampere-turns per centimetre, and when it is 200 ampere-turns per inch.

(8) How many ampere-turns are required for one C.G.S. unit of excitation per cm. length?

(9) Determine (i) the M.M.F. (in C.G.S. units) of the excitation due to 100 amperes traversing a spiral of 10 turns to the inch.

(ii) The M.M.F. in ampere-turns required to produce an intensity of field of 200 units within a helix 25 centimetres long.

(10) You have to make a long coil of wire such that when 10 amperes flow through it the magnetic field at the centre shall be 1000 times as strong as the earth's horizontal field (H). How many turns per centimetre must be put on the coil? $H = 0{\cdot}18$. (C. and G. 91.)

(11) You are required to wind over a brass tube 500 centimetres long and 2 centimetres external diameter with one layer of covered wire, 1 mm. diameter over the covering. What length of wire will you require? What will be the strength of the magnetic field at the centre of the axis of such a helix when a current of one ampere flows through the wire? (C. and G. 87.)

(12) Determine the intensity of the magnetic field at the centre of a solenoid 1 foot in length, consisting of 2400 turns, if 0·5 ampere traverses the coil. How many lines per square inch will there be at the centre of this coil?

(13) Determine the current required to produce 1500 lines per square inch in a solenoid 15 inches long and consisting of 1400 turns.

(14) Calculate approximately the strength of the magnetic field produced at the centre of a solenoid 4 inches long, the coils of which are $1\frac{1}{2}$ inches thick and have a mean diameter of $7\frac{1}{2}$ inches, when the current density taken over wire and insulation together and measured on a plane containing the axis of the solenoid is 750 amperes per square inch. Could a coil so constructed and working with this current density be used for long periods without due heating? (C. and G.)

(15) The primary winding of a transformer in the form of a ring, 20 centimetres mean diameter, consists of 450 turns, and carries 7·5 amperes. Determine the magnetizing force and the magnetomotive-force of the coil.

Section II. Magnetization of Iron.

§ 42. Magnetization of Iron. Induction. Permeability. Susceptibility. In the preceding paragraphs we have only considered the magnetic field inside a hollow helix, such as is represented by H in Fig. 33, and when we remember that unmagnetized iron and steel become magnetized by induction if placed in a magnetic field, it is evident why H is known as the magnetizing force as well as field intensity, inasmuch as it is the measure of the force which sets up or tends to set up the magnetization in the iron. As the result of the magnetization of the iron by induction, the actual magnetic condition

of the iron or steel, known as the *magnetic induction* or simply the *induction*, is found by adding the number of lines of force of the field per square centimetre, or H, to the $4\pi I$ lines of magnetization of the iron, the sum of which is denoted by the letter B. B therefore denotes that magnetic condition termed the *induction*, which is graphically represented at B, Fig. 33. Thus

$$B = H + 4\pi I$$

where B = the total number of lines of force per square centimetre, or the *flux-density*, now passing through the iron or steel. A magnetic force H thus produces a flux-density of H lines of force per square centimetre in air or other non-magnetic substance, and a *flux-density* of B lines of force per square centimetre in iron or steel; in other words, the presence of iron in a magnetic field increases the number of lines of force by $(B - H)$ lines per square centimetre, where $B = \mu H$, μ denoting the multiplying power of the iron, B being often one thousand or more times H. It is clear then that induction is an *effect* of which magnetic force is the *cause*. The effect of placing a magnetic substance in a magnetizing coil is twofold.

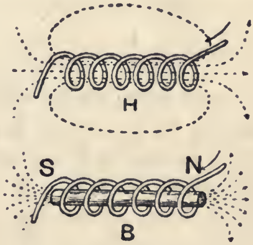


Fig. 33.

1. The lines of force are concentrated, iron being more *permeable* to lines of force than air; and

2. The number of the lines of force are increased by reason of the iron being magnetized by induction.

The drawing together or focussing of the lines of force of magnetic substances is due to the superior *magnetic conductance* or *permeance* of these substances as compared with that of air. The relative conducting power of iron for lines of force as compared with that of air is expressed by the value of the ratio of B to H or μ, since

$$B = \mu H \text{ and } \frac{B}{H} = \mu = \frac{\text{flux density}}{\text{magnetic force}}.$$

The value of this ratio, or μ, is known as the *magnetic permeability* of the iron corresponding to that particular flux density, B, and for good soft wrought iron μ may have a value as high as 2000 or 3000. From the equation, $B = \mu H$, it is evident that μ, or magnetic permeability, is of the nature of a numerical coefficient, and denotes the *magnetic conductivity* of the iron or steel. Since $B = H$ when a magnetic substance is not present, the permeability of air is said to be unity, and air is therefore taken as the standard of reference. Experimentally it is found that permeability is a variable quantity,

and depends upon the degree of magnetization, the value of μ changing with every change of H within certain limits. Permeability also varies with different samples of iron or steel, and depends largely upon the quality of the material. Permeability is thus a physical property, and is an important factor to be considered in dynamo and transformer design.

For some purposes it is useful to know the value of the ratio of the intensity of magnetization, I, to the magnetic force, H, and where I is entirely induced by H, the ratio of I to H, or $\frac{I}{H}$, is known as the *magnetic susceptibility* of the material, and is denoted by the letter κ; therefore $\frac{I}{H} = \kappa$. Since the magnetic character of a substance is known when either of the quantities, μ or κ, is determined, it will be useful to determine the relationship between these two quantities. This is readily found as follows:—

Since $B = \mu H$ and $I = \kappa H$

and $B = H + 4\pi I$

we have $\mu H = H + 4\pi\kappa H = H(1 + 4\pi\kappa)$

or $\mu = 1 + 4\pi\kappa$

from which we get $\kappa = \frac{\mu - 1}{4\pi}$

§ **43. Reluctance.** From the fact that permeability varies with different qualities of iron, it follows that different substances conduct magnetic flux in different degrees for the same magnetizing force, and we are led to the conclusion that a property, analogous to electrical resistance, is a factor in all magnetic circuits. That the conception of magnetic resistance is a useful one will be apparent from the following considerations. Let us suppose that a complete iron ring of circular cross section—termed an anchor-ring or toroid—be wound over closely with a spiral coil of insulated wire through which a current of electricity is passing as shown in Fig. 34. As with a straight coil and bar the spiral current sets up a state of magnetization in the iron ring, and the resulting induction, B, is perfectly definite both as to intensity and direction (indicated by the dotted lines in the figure), since the ring is traversed by a definite magnetic flux, although as a whole it exhibits no magnetic polarity or other external magnetic effects. This is what would be expected, inasmuch as the iron core is endless.

If the anchor-ring, however, be cut through at any one place (Fig. 35) so as to leave two ends separated by a very narrow air gap or crevasse (assumed to be so narrow that the uniformity of the

magnetization is not disturbed), then the iron core will have two polar surfaces between which a definite magnetic field is produced, and the secondary effects of polarity, attraction and repulsion follow

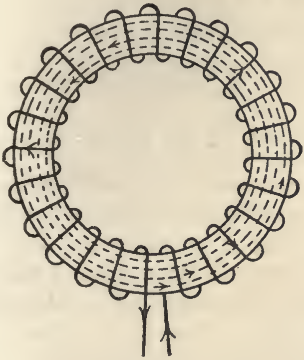


FIG. 34.

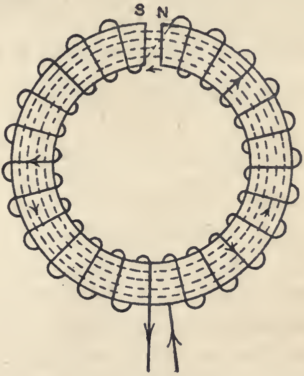


FIG. 35.

the formation of the poles, although the magnetization existed previously.

Now, with the undivided anchor-ring we have the simplest and most perfect of magnetic circuits, and its magnetic qualities are easily determined when C amperes traverse the S turns of wire.

Let A = area of the cross section of the iron in square centimetres;
r = mean radius of the ring in centimetres.
μ = the permeability of the iron for the excitation of CS ampere-turns.
N = total magnetic flux traversing the iron
$= \mathsf{BA}$;

$$\text{then } \mathsf{H} = \frac{0{\cdot}4\pi CS}{l} = \frac{1{\cdot}257\,CS}{l} \text{ where } l = 2\pi r$$

$$= \frac{0{\cdot}4\pi CS}{2\pi r} = \frac{0{\cdot}2\,CS}{r};$$

$$\text{and } \mathsf{B} = \mu\mathsf{H} = \frac{0{\cdot}4\pi CS \times \mu}{l} = \frac{1{\cdot}257\,CS\mu}{l};$$

$$\text{and } \mathsf{N} = \mathsf{BA} = \frac{1{\cdot}257\,CS\mu A}{l}$$

$$= \frac{1{\cdot}257\,CS}{\frac{l}{A} \times \frac{1}{\mu}} = \frac{\text{M.M.F.}}{\frac{l}{A} \times \frac{1}{\mu}};$$

$$\therefore \text{(total) flux} = \frac{\text{magnetomotive-force}}{\frac{l}{A} \times \frac{1}{\mu}}$$

If we write k for $\frac{1}{\mu}$ it is obvious that $\frac{l}{A} \times \frac{1}{\mu}$ or $\frac{kl}{A}$ is identical with the formula for calculating the electrical resistance of a conductor. For this reason the quantity $\frac{l}{A} \times \frac{1}{\mu}$ is termed the magnetic resistance or *reluctance* of the magnetic circuit, and $\frac{1}{\mu}$, the reciprocal of the magnetic permeability or conductivity, denotes the *magnetic reluctivity* (or *specific magnetic resistance*) of the medium forming the magnetic circuit. Reluctance is thus the reciprocal of permeance, and is the resistance offered to the passage of magnetic flux due to the magneto-motive-force acting on the circuit, and like electrical resistance it limits the magnitude of the magnetic flux. This term, reluctance, was introduced by Mr. Oliver Heaviside, and is now commonly used. Since the reluctivity of a medium or substance is defined as *the magnetic resistance offered to the passage of magnetic flux between two opposite parallel faces of an isolated centimetre cube of the substance*, the value of the reluctance of a circuit is readily found as follows:—

$$\text{Reluctance} = \frac{\text{reluctivity} \times \text{length (cms.)}}{\text{area (sq. cms.)}}$$
$$= \frac{\text{length (cms.)}}{\text{permeability} \times \text{area (sq. cms.)}}$$

If $\mathfrak{R}$ = the reluctance of a magnetic circuit

$$\mathfrak{R} = \frac{1}{\mu} \times \frac{l}{A}$$

Again, since reluctivity is the reciprocal of permeability, the reluctivity of air and non-magnetic substances is unity, and that of magnetic substances $\frac{1}{\mu}$, or a value less than unity, there is no substance possessing an infinitely high magnetic resistance, nor is there any insulator of magnetic flux, for all known substances conduct magnetic flux to some extent. It is thus clear from what has already been said, that a magnetic circuit is analogous to an electric circuit, and Ohm's Law for a magnetic circuit is as follows:

$$\text{Magnetic flux} = \frac{\text{excitation}}{\text{reluctance}}$$

or

$$N = \frac{\text{M.M.F.}}{\mathfrak{R}} \text{ and M.M.F.} = N \times \mathfrak{R}$$

or magnetomotive force = magnetic flux × reluctance, which is the analogue of the equation, $E = C \times R$, of the electric circuit. Now, although this similarity exists between magnetic and electric circuits, there are certain distinct points of dissimilarity existing between them which it will be well to briefly notice. On the one hand, the resistivity of a material is a constant quantity, whilst the reluctivity of magnetic materials is a variable quantity. Electrical resistance, for instance, is a physical property of the material itself, and in no case is it a function of the current traversing it, being, in fact, a constant quantity so long as the temperature remains constant; whilst, as we have seen, magnetic resistance or reluctance is a function of the flux density and is not a constant quantity, and never depends solely upon the nature of the material.

On the other hand, whilst both electric and magnetic fluxes are intimately associated with energy, and the existence of resistance is associated with the dissipation of energy, it is important to notice that a constant expenditure of $C^2 R$ units of energy takes place so long as the current is maintained in an electric circuit, but that in a magnetic circuit there is no expenditure of energy required to maintain the existence of a magnetic flux after once it has been established, although an amount of energy equal to $\frac{B^2}{8\pi}$ ergs per cubic centimetre is stored up in a magnetic flux, of flux density, B, so long as the flux is maintained. It is, of course, obvious that a definite amount of energy must be expended to establish a magnetic flux, since work is done during the process of magnetization, and energy must be imparted to a magnet or magnetic field in producing either of them.

§ 44. **Measurement of Magnetic Flux and Magnetic Force.** Physical quantities are usually measured by some physical effect intimately connected with the quantity to be measured; thus, in the case of the quantities magnetic flux and magnetic force, advantage is taken of the discovery made by Faraday that an induced electromotive-force is set up in a second coil of wire—termed a secondary or exploring coil—wound round a ring, around which a current-carrying coil already exists, whenever a change occurs in the flux traversing the ring. And, since the electromotive-force, due to the introduction, removal, reversal or change of any kind in the flux, is proportional to the *time rate of change* in the flux, the total induction through any area may be practically measured by connecting the ends of the secondary coil (enclosing the area) to the terminals of a ballistic galvanometer, which enables us to measure the quantity of electricity thereby impelled through the secondary circuit and galvanometer. If there be n turns of wire forming the secondary

coil the induction is then equal to the quantity of electricity so impelled, divided by n times the resistance of that circuit. Thus, if the quantity of electricity impelled through the galvanometer be n coulombs and the resistance of the secondary circuit be one ohm, the *induction will be one hundred million maxwells.* Or, if the induction through a single turn of wire change at the rate of one hundred million maxwells per second, the electromotive-force set up in that coil will be one volt. Hence the following definition :—

The Maxwell, the unit of magnetic flux, is equal to one C.G.S. line of induction, and is such that if a flux of 10^8 *maxwells be uniformly withdrawn from a co-linked secondary circuit of one turn in one second, an electromotive-force of one volt is set up in that circuit.*

It may be remarked here that names have not yet been given to all the electromagnetic units (see p. 331); as we have previously mentioned another unit, that of *field-intensity* and *magnetizing force*, has been named and is termed the *Gauss.* The unit of flux-density or B (lines per square centimetre in iron) is the *maxwell per square centimetre.*

In the C.G.S. system of units unit intensity in air (i. e. one C.G.S. line of force per square centimetre in air) is produced by the C.G.S. unit of magnetic force, or the *Gauss,* and the whole magnetomotive-force (M.M.F.) of a magnetic circuit is the *gaussage* of that circuit multiplied by the length.

Since $$\mathsf{B} = \mathsf{H} \text{ (in air) and } \mathsf{H} = \frac{1{\cdot}257\,Cn}{l}$$

and one ampere-turn per centimetre sets up 1·257 lines of force in air,

$\therefore$ $\dfrac{1}{1{\cdot}257}$ or 0·8 of an ampere-turn per centimetre sets up unit intensity in air, but the magnetic force of one gauss sets up unit intensity in air,

$\therefore$ one gauss = 0·8 of an ampere-turn per centimetre length.

Consequently, if the earth's horizontal intensity be 0·18 C.G.S. unit at a certain place, we should say that the magnetic force (horizontally) at that place is 0·18 gauss. Similarly, the circuital gaussage round a closed magnetic circuit is $\dfrac{4\pi}{10\,l}$ times the ampere-turns of excitation.

§ 45. **Experimental Determination of Magnetic Properties of Iron and Steel.** Since the magnetic properties of iron and steel play so important a part in the working of dynamo-electric machinery, and bearing in mind the variable and complex nature of permeability, it is obviously essential that the process of magnetization be experimentally investigated, and the relationships between the quantities induction (B) and magnetic force (H), permeability (μ) and induction

(B), and permeability (μ) and magnetic force (H) be determined for all the different samples of iron and steel used in the construction of such machinery, both under weak and strong magnetic fields. The fact is so well recognized that many investigators have devised methods for making these determinations, but more cannot be said here than to refer briefly to the most common one, i.e. the ballistic method. In this method a complete ring of the sample to be tested is wound with two coils of wire, one, the magnetizing coil (or primary), completely surrounds the ring as previously described, and the other (consisting of a few turns of fine wire, and termed the secondary or exploring coil) is also wound for a short distance on the ring (preferably under the primary or magnetizing coil), and the ends of this coil are connected to the terminals of a ballistic galvanometer. We shall assume that at the beginning of the test the iron forming the ring is in the neutral state, and has been previously softened by annealing; also, that means exist for gradually increasing the magnetizing force. As we have already explained, a magnetic flux traverses the ring whenever a current traverses the magnetizing coil, and the intensity of the flux changes with every change in the current strength, but not proportionately. We have already stated that when any sudden variation in the flux takes place a transient current is produced in the exploring coil, and gives rise to a corresponding swing of the galvanometer needle. And since the swing of a slow swinging galvanometer needle, such as that of a ballistic galvanometer, is proportional to the change in the total flux in the iron ring, and therefore to the induction (B), when the changes are made so quickly that the needle has not begun to swing before the variation of the flux density has finished, it is an easy matter to give a quantitative meaning to the readings of the galvanometer by calibrating the galvanometer beforehand. Therefore, by applying gradually increasing magnetic forces to the ring we obtain measurements of the effects produced in the flux density, and these may be expressed in the form of a table or curve.

§ 46. **Results of Tests. Magnetization and Permeability Curves.** Probably the most useful way of representing the magnetic qualities of different samples of iron and steel is to plot the results obtained experimentally on squared paper. The characteristic curve given by the diagram formed by taking lengths on a horizontal axis to represent H or magnetic force, and lengths on a vertical axis to represent the corresponding magnetic flux density, or B, is called a *magnetization curve.* The magnetization curve for a sample of wrought-iron is given in Fig. 36, from which it is obvious that the form of the curve indicates three well-defined stages in the

process of magnetization. In the first stage an increase in the magnetizing force produces (approximately) a proportional increase in the flux density, indicating that at this portion of the process of magnetizing iron, the permeability is independent of the magnetic force. In this the *initial stage*, the proportionality between B and H only holds for very small values of H, the limit being reached when H equals about one C.G.S. unit. After that value of the magnetic force is reached, i. e. in the *second stage*, a high value of the induction is obtained with a small increase in the magnetizing force, the curve being nearly straight and rising steeply to a *knee*, indicating that the increase in the flux density is extremely rapid. The knee or

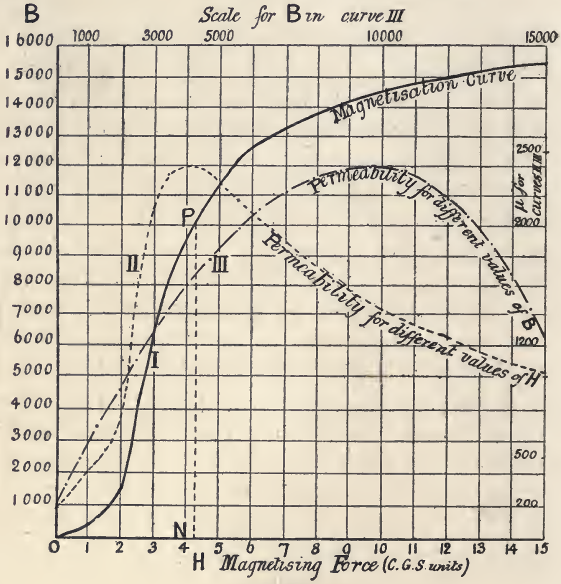


FIG. 36.

bend in the curve corresponds to about 4 C.G.S. units of the magnetizing force (H), after which we have the slow *final stage*, in which further increase in the magnetizing force produces only a very small increase in the induction, and the curve is now inclined at a small angle to the horizontal axis. The iron is then said to approach *saturation*. The first of these stages has another important characteristic, i. e. if the magnetizing force has not been carried beyond the limits corresponding to the first stage, no magnetism can be detected when the magnetizing force is removed; in other words, the iron is not permanently magnetized. This therefore may be called the stage of perfect magnetic elasticity.

If any point P be taken on the curve (Fig. 36) then the ordinate PN denotes the induction B corresponding to the magnetic force H denoted by the abscissa ON, and the value of the permeability, &c., for that induction is given by the ratio

$$\frac{\text{ordinate PN}}{\text{abscissa ON}} = \frac{B}{H} = \mu.$$

The relationships between μ and H, and between μ and B, are given by the curves II and III in the same figure. These are known as permeability curves, and indicate very clearly the variations in the value of the permeability corresponding to different values of H and B.

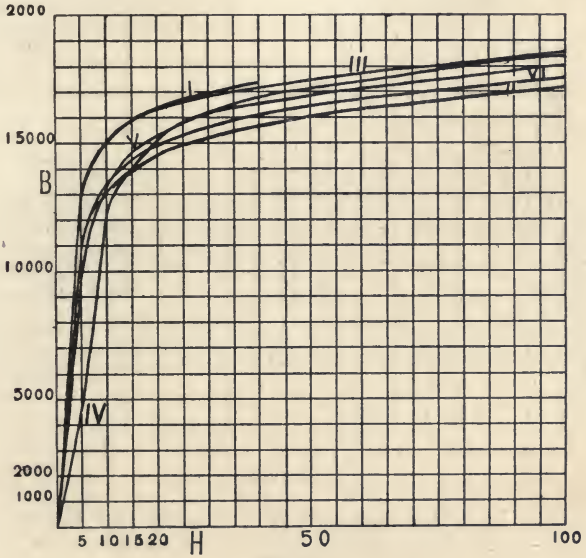


Fig. 37.

At this point it may be mentioned that the magnetic properties of iron and steel are largely affected by their chemical and physical properties, and it is important to notice that the general form of the magnetization curve for any material is characteristic of the class of material, and is indicative of the special duty for which it is most suitable. Consequently the comparison of the characteristic curves enables the engineer to compare the magnetic qualities of the materials from which they have been obtained. The table I of numerical results and the characteristic curves of six samples of iron and steel (Figs. 37 and 38), all tested in the form of solid turned rings and using the ballistic method, are given as examples of the practical determination of magnetization curves. These results are taken from

Prof. Ewing's paper on 'The Magnetic Testing of Iron and Steel,' read before the Institution of Civil Engineers, May 19, 1896.

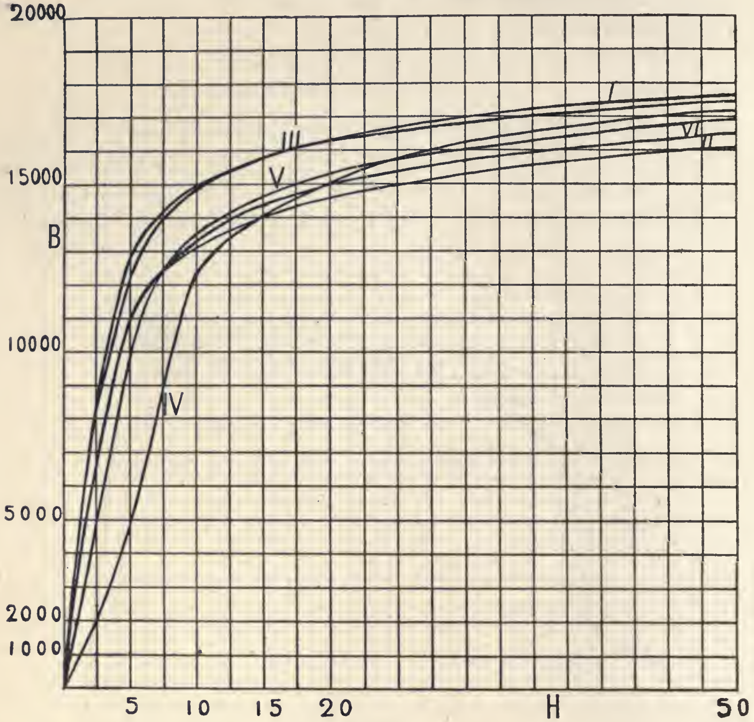


FIG 38.

TABLE I.—PERMEABILITY OF FORGINGS AND CASTINGS.

Magnetizing Force H.	I.		II.		III.		IV.		V.		VI.	
	B	μ	B	μ	B	μ	B	μ	B	μ	B	μ
5	12,700	2,540	10,900	2,180	12,300	2,460	4,700	940	9,600	1,920	10,900	2,180
10	14,980	1,498	13,120	1,312	14,920	1,492	12,250	1,225	13,050	1,305	13,320	1,332
15	15,800	1,054	14,010	934	15,800	1,054	14,000	933	14,600	970	14,350	956
20	16,300	815	14,580	729	16,280	814	15,050	752	15,310	765	14,950	747
30	16,950	565	15,280	509	16,810	560	16,200	540	16,000	533	15,660	522
40	17,350	434	15,760	394	17,190	429	16,800	420	16,510	413	16,150	404
50	—	—	16,060	321	17,500	350	17,140	343	16,900	338	16,480	329
60	—	—	16,340	272	17,750	296	17,450	291	17,180	286	16,780	279
70	—	—	16,580	237	17,970	257	17,750	253	17,400	250	17,000	244
80	—	—	16,800	210	18,180	225	18,040	226	17,620	220	17,200	215
90	—	—	17,000	188	18,390	204	18,230	203	17,830	198	17,400	193
100	—	—	17,200	172	18,600	186	18,420	184	18,030	180	17,600	176

No. I. was a sample of exceptionally pure iron; No. II. was a sample of Low Moor bar, forged into a ring, annealed and turned; No. III., a steel forging for dynamo magnets; No. IV., steel casting, unforged; No. V., unforged steel casting; and No. VI., unforged steel casting.

Worked Examples. (1) Determine the reluctance and permeability of a wrought iron ring 100 cms. mean circumference and 5 sq. cms. cross section if the magnetic flux is 65000 lines, given that the magnetizing force is 922 ampere-turns.

Since
$$N = B \times A = \mu HA$$
$$= \frac{1.257\ \text{CSA}\,\mu}{l}$$
$$\mu = \frac{Nl}{1.257\ \text{CSA}}$$
$$= \frac{65000 \times 100}{1.257 \times 922 \times 5} = 1121.7$$

Again
$$\mathcal{R} = \frac{l}{A} \times \frac{1}{\mu}$$
$$= \frac{100}{5} \times \frac{1}{1121.7}$$
$$= 0.0178$$

(2) A closed wrought iron ring 100 cms. mean circumference and 4 sq. cms. cross sectional area is uniformly wound with 250 turns of insulated wire. If the following relations exist in iron of the quality used :

$B = 11000$	12000	13000
$\mu = 1692$	1412	1083

Calculate the current required to give a total flux of 50000 lines.

Since $B = \mu H$, it is an easy matter to find the magnetizing forces H which give the above values of the flux-densities, thus

$$\text{for } B = 11000 \qquad H = \frac{11000}{1692} = 6.5$$
$$B = 12000 \qquad H = \frac{12000}{1412} = 8.5$$
$$B = 13000 \qquad H = \frac{13000}{1083} = 12.0$$

By plotting the corresponding values of B and H, the part of the magnetization curve used for the case under consideration (see Fig. 41) is obtained; and the value of H required for a flux of 50000 lines may be obtained directly therefrom.

The flux density for the given flux is
$$B = \frac{50000}{4} = 12500$$
and by dropping a perpendicular from the point on the curve, Fig. 41, giving $B = 12500$, the corresponding value of H is found to be 10;

but
$$H = \frac{1.257\ \text{CS}}{l} \qquad \therefore\ C = \frac{H \times l}{1.257 \times S}$$
$$\text{and } C = \frac{10 \times 100}{1.257 \times 250} = 3.18 \text{ amperes.}$$

EXERCISES V B.

Magnetic Circuit. I.

(1) Determine the value of the reluctance of a bar of iron, 150 centimetres long and 5 square centimetres in cross section, when μ for the excitation used is 400.

(2) Find the magnetic reluctance and permeability of a wrought iron bar, 25 centimetres long and 2·5 square centimetres in cross section, if the magnetic flux is 56565 lines, given that the magnetizing force in ampere-turns is 500.

(3) Find the area of cross section of a bar of iron 100 cms. long, of which the reluctance is 0·08 when $\mu = 300$.

(4) Determine the permeability of a sample of iron when the reluctance of a ring of iron is 0·075, given that the mean circumference of the ring is 50 cms., and the area of cross section 2 sq. cms.

(5) The core of an armature is 14 cms. long and the depth of the iron is 6 cms. If the permeability of the iron is 750 and the length of the path traversed by the flux is 15 cms., determine the reluctance. If the total flux is 1000000 lines what M.M.F. does this correspond to?

(6) A helix 1 metre long, with an area of cross section of 1 sq. cm., consists of 770 turns of wire. If a current of 2 amperes is sent through it, determine the number of lines of force passing through a certain sample of iron of permeability 750, which just fits the helix.

(7) A closed soft iron ring, 100 cms. mean circumference and 5 sq. cms. cross section, is uniformly wound with 200 turns of insulated wire. Suppose you have found that the following relations exist in iron of this quality:

$B =$	10200	12000	13700
$\mu =$	2000	1500	1000

Calculate the current C at which the total flux of magnetic lines is 65000 C.G.S. lines. (C. and G. 92.)

(8) A soft iron ring the mean diameter of which is 10 cms. is wound with 160 turns of wire which when traversed by 2·75 amperes produces a flux of 35000 lines. If the area of cross section is 2·5 sq. cms. determine the permeability.

(9) Find the number of ampere-turns of excitation required to magnetize up to 14000 lines per sq. centimetre a soft iron ring 50·8 cms. in mean diameter, made of round iron 2·54 cms. thick. Permeability = 1778.

(10) The iron core of a transformer is excited so as to produce flux densities of 5000, 10000 and 12500 lines respectively. The cross-

sectional area is 50 sq. cms., and the length of the magnetic circuit 1000 cms. ; determine the reluctance in each case, if the magnetic properties of the iron are those given in table II, p. 208. If there are 500 turns of wire on the primary coil what current will be required to produce the flux-density in each case?

(11) An iron ring 10 square centimetres cross-sectional area is wound with 432 turns of wire carrying 2 amperes, and is traversed by 500000 lines of force. Determine the permeability of the iron if the mean circumference of the ring is 91·5 centimetres.

(12) It is required to produce 3600000 lines of force in a bar of wrought iron 125 cms. long and 38·7 sq. cms. cross-sectional area. If the permeability of the iron is 332 determine the excitation in ampere-turns required.

(13) Determine the reluctance of the magnetic circuit in Exercise 12.

(14) Given that the following relations exist in transformer iron:

B =	5000	10000	14000
μ =	2500	2000	1000

determine the magnetic reluctance of the core of a transformer 100 cms. in length and 40 sq. cms. cross-sectional area when the flux-density is respectively 5000, 10000 and 14000.

(15) Determine the excitation in ampere-turns required to produce the respective inductions in Exercise 14.

(16) What is the moment of a bar magnet 10 cms. long and 2·5 sq. cms. in cross-sectional area, if the intensity of magnetization is 150 C.G.S. units?

(17) A long bar magnet 4 cms. by 2 cms. in cross section has a pole strength of 2240 C.G.S. units, how many lines of force issue from the N pole per sq. cm., and what is its intensity of magnetization?

(18) Assuming the specific gravity of a steel bar magnet to be 7·85 and its weight 453·6 grammes, and the intensity of magnetization 174·6 C.G.S. units, find the pole strength if the magnet is 50 cms. long.

(19) What is the number of lines of force per sq. cm. which issue from a bar magnet whose intensity of magnetization is 1000 when the magnetizing force is 50 units?

(20) The susceptibility of a certain bar of soft iron was 200, determine the permeability of the bar.

(21) A steel bar 20 cms. long and 2·5 sq. cms. in cross section was placed between the poles of an electromagnet where the field was uniform and of 20 units strength. If the permeability of the bar was 700, what was the resulting magnetic moment and intensity of magnetization?

(22) A magnet 20 cms. long and 0·8 cm. by 0·5 cm. has a magnetic

moment of 1500 units. The magnet weighs 40 grammes. Determine (1) the moment per gramme, (2) intensity of magnetization, and (3) pole strength.

(23) A straight piece of soft iron wire 100 cms. long and 0·1 cm. in diameter is placed horizontally in the magnetic meridian, where $H = 0{\cdot}19$ determine the intensity of magnetization if the coefficient of susceptibility of the iron is 40, and also its magnetic moment.

(24) Calculate approximately the strength of the magnetic field produced at the centre of a solenoid 4 inches long, the coils of which are $1\frac{1}{2}$ inches thick and have a mean diameter of $7\frac{1}{2}$ inches, when the current density taken over wire and insulation together and measured on a plane containing the axis of the solenoid is 750 amperes per sq. inch. (C. and G.)

(25) An electromagnet has to be designed to produce an approximately uniform field of 10000 lines per square centimetre over an area of 12 sq. centimetres in an air gap half a centimetre long. Sketch an electromagnet, approximately to scale, with which this result can be obtained without undue heating of the coils. State the gauge of wire, number of convolutions, and current that may be used, and describe in detail how your results are obtained. (C. and G.)

Section III. Magnetic Circuit of a Dynamo.

§ 47. Compound Magnetic Circuits. In practice the magnetic circuits are of a compound character, and, in the case of dynamo machinery, the core of the armature, air gaps, and the core of the field magnets and yoke compose the complete magnetic circuit. It is therefore obvious that the algebraic statement which we have previously given for Ohm's Law of the magnetic circuit must be modified so as to include the reluctances of the separate portions, and the result is a more complex formula than the one already given. For the present we shall assume that the length of the air gap is not sufficient to produce magnetic leakage, but later on it will be necessary to consider such effects. The simplest case of a compound magnetic circuit containing an air gap is that of a ring of iron in which a saw cut is made so as to leave two ends separated by a very narrow gap or crevasse (assumed to be so narrow that the uniformity of the magnetization is not disturbed), then the iron core will have two polar surfaces between which a definite magnetic field is produced, and the secondary effects of polarity, attraction, and repulsion follow the formation of the poles, although the magnetization existed previously. When the iron core (with an air gap) is excited by a

current circulating through the coils, the magnetic flux passes through the iron and is continued through the air gap. The reluctance, however, of the circuit is increased, and the magnitude of the resulting flux is diminished. Neglecting leakage we have for the simple magnetic circuit shown in Fig. 35 the total magnetic flux, N, given by the equation

$$\mathsf{N}\ (\text{total flux}) = \frac{1{\cdot}257\,\mathrm{CS}}{\dfrac{l_1}{A_1\mu_1} + \dfrac{l_2}{A_2\mu_2}}$$

where l_1 and l_2 are the lengths of the air gap and iron core

A_1 and A_2 the corresponding areas of cross section,

μ_1 and μ_2 the permeabilities of the air and iron respectively.

If r = the mean radius of the ring, this formula reduces to

$$\mathsf{N} = \frac{1{\cdot}257\,\mathrm{CS}}{\dfrac{l_1}{A_1} + \dfrac{2\pi r - l_1}{A_2\mu_2}}$$

Since μ_1, for air, is unity.

Worked Example. An iron ring 5 sq. cms. cross section and 10 cms. mean radius has an air gap ·5 mms. wide cut across it. The ring is wound uniformly with a number of turns, so that when 5 amperes are sent through the wire the strength of the resulting field in the gap is 10000 C.G.S. units. Given that the permeability of the iron is 2000 for the excitation used, determine the number of turns.

$$\mathsf{N} = \frac{1{\cdot}257\ \mathrm{CS}}{\dfrac{l_1}{A_1\mu_1} + \dfrac{l_2}{A_2\mu_2}} = \frac{1{\cdot}257\ \mathrm{CS}}{\dfrac{l_1}{A_1\mu_1} + \dfrac{2\pi r - l_1}{A_2\mu_2}}$$

and $\mathrm{C} = 5$ amperes, $\mathsf{N} = 10000 \times 5$, $l_1 = {\cdot}5$ cm., $l_2 = (2\pi \times 10 - {\cdot}5)$,
$A_1 = A_2 = 5$ sq. cms., $\mu_1 = 1$ and $\mu_2 = 2000$.

$$\therefore \quad 10000 \times 5 = \frac{1{\cdot}257 \times 5 \times \mathrm{S}}{\dfrac{{\cdot}5}{5} + \dfrac{20\pi - {\cdot}5}{5 \times 2000}}$$

$$\therefore \quad \mathrm{S} = \frac{10000 \times 5}{5 \times 1{\cdot}257}\left(\frac{{\cdot}5}{5} + \frac{62{\cdot}832 - {\cdot}5}{5 \times 2000}\right)$$

$$= \frac{2000}{1{\cdot}257}\left({\cdot}5 + \frac{62{\cdot}332}{2000}\right)$$

$$= \frac{2000}{1{\cdot}257}({\cdot}5 + {\cdot}031166)$$

$$= \frac{2000 \times {\cdot}531166}{1{\cdot}257}$$

$$= 845 \text{ turns.}$$

§ 48. Magnetic Circuit of the Dynamo. Hopkinson's Method. From the equation of a compound magnetic circuit considered in the

last example, it is obvious that for a complete magnetic circuit of a dynamo we shall have

$$N = \frac{1{\cdot}257\,CS}{\dfrac{l_a}{A_a\mu_a} + \dfrac{l_g}{A_g\mu_g} + \dfrac{l_m}{A_m\mu_m} + \dfrac{l_y}{A_y\mu_y}}$$

in which

l_a, l_g, l_m, l_y represent the lengths of the armature, air gap, magnet cores, and yokes respectively,

and

A_a, A_g, A_m, A_y, and μ_a, μ_g, μ_m, μ_y represent the areas of cross section and permeabilities of the several portions of the circuit respectively.

Now, from the analogy of the electric circuit it is clear that the total magnetizing influence, 1·257 CS, is the sum of the separate magnetizing influences required to overcome the individual reluctances of the parts. Thus, there is required for the armature portion a magnetizing influence of $N \times \dfrac{l_a}{A_a\mu_a}$ units. Denote this by Ω_a, and by symmetry Ω_g, Ω_m, and Ω_y will denote the excitation required by the portions, the air gap, the field magnet cores, and yoke respectively. Let Ω similarly denote the M.M.F. (magnetomotive-force) of the complete circuit. Then

$$1{\cdot}257\ CS = \Omega = \Omega_a + \Omega_g + \Omega_m + \Omega_y.$$

And, as in the electric circuit a specific difference of potential, P.D., has to be maintained between any two points in the circuit, so in the magnetic circuit there must be a definite difference of magnetic potential (say, H^1,) for each centimetre length of the circuit, and provided that the induction, B, be the same for a length, l, then the difference of magnetic potential required to send the flux-density, B, l cms., or the fall of magnetic potential over a length, l, is

$$H^1 \times l = \Omega_1\,(\text{say}) = \frac{N_1\,l}{A_1\,\mu_1} = \frac{B_1}{\mu_1} \times l$$

In other words, the fall of magnetic potential over a length of magnetic circuit, or the difference of magnetic potential required for a length of l cms., is the same thing as the excitation or magnetizing influence required for that portion of the circuit. These results indicate a method of finding the total excitation required for the complete magnetic circuit of a dynamo, and the most convenient method is to calculate the magnetizing influence required for each separate part, and then add these together. For practical purposes it is advisable to give the M.M.F., or the difference of magnetic potential required for any portion, simply in ampere-turns, instead of

as 1·257 CS or 0·4 π CS. We shall therefore now show how to modify the general expressions given above, and to give the result in ampere-turns.

For a simple magnetic circuit

$$N = \frac{\text{M.M.F.}}{\text{reluctance}} = \frac{1{\cdot}257\,\text{CS}}{\frac{l}{A} \times \frac{1}{\mu}}$$

and
$$\text{M.M.F.} = 1{\cdot}257\ \text{CS} = \frac{N}{A} \times \frac{l}{\mu}$$

$$= \frac{B}{\mu} \times l = H \times l$$

from which it is obvious that the magnetizing force is some function of the induction, therefore $\frac{N}{A} \times \frac{1}{\mu}$ is some function of B expressed in terms of H. The symbolical expression for the function of a quantity, x, is f(x), therefore the excitation may be expressed as

$$\text{M.M.F.} = f\left(\frac{N}{A\,\mu}\right) \times l$$

$$= f(B) \times l$$

Now, since
$$H = \frac{1{\cdot}257\,\text{CS}}{l} = 1{\cdot}257\,\text{Cs}^1$$

where
$$s^1 = \frac{S}{l} = \text{number of turns per centimetre length}$$

$\therefore$ H = 1·257 times the ampere-turns per cm. length.

and
$$\text{Cs}^1\ (\text{per cm. length}) = \frac{H}{1{\cdot}257} = 0{\cdot}8\ H$$

$$= f(B)\ [\text{function of } B]$$

and the total number of ampere-turns required for the simple circuit of length l is

$$\text{CS} = f(B) \times l = 0{\cdot}8\ H \times l$$

To take advantage of this method the values of $f(B)$ or 0·8 H for the various inductions, B, are calculated and inserted in the tabulated sets of observations made on the magnetic properties of the materials used; thus a fourth column is added in the table on p. 208. We may now give the modification of the first equation given above, from which it will be seen that μ need not be introduced into the calculations of the magnetic circuit of a dynamo. Thus we have

$$N = \frac{1{\cdot}257\ \text{CS}}{\frac{l_a}{A_a\,\mu_a} + \frac{l_g}{A_g\,\mu_g} + \frac{l_m}{A_m\,\mu_m} + \frac{l_y}{A_y\,\mu_y}}$$

$$\text{and } 1{\cdot}257\ \mathrm{CS} = \mathrm{N} \times \left(\frac{l_a}{A_a \mu_a} + \frac{l_g}{A_g \mu_g} + \frac{l_m}{A_m \mu_m} + \frac{l_y}{A_y \mu_y}\right)$$

$$\begin{aligned}\therefore\ \mathrm{CS} &= \frac{\mathrm{N}}{1{\cdot}257}\left(\frac{l_a}{A_a \mu_a} + \frac{l_g}{A_g \mu_g} + \frac{l_m}{A_m \mu_m} + \frac{l_y}{A_y \mu_y}\right)\\ &= f(\mathrm{B}_a) \times l_a + f(\mathrm{B}_g) \times l_g + f(\mathrm{B}_m) \times l_m + f(\mathrm{B}_g) \times l_y\\ &= 0{\cdot}8\ \mathrm{H}_a\, l_a + 0{\cdot}8\ \mathrm{H}_g\, l_g + 0{\cdot}8\ \mathrm{H}_m\, l_m + 0{\cdot}8\ \mathrm{H}_y\, l_y\\ &= \Omega_a + \Omega_g + \Omega_m + \Omega_y \text{ in ampere-turns.}\end{aligned}$$

The following table gives the result of Dr. Hopkinson's tests on annealed wrought iron, to which values of $f(\mathrm{B})$ have been added :—

TABLE II.—ANNEALED WROUGHT IRON.

B	H	$\mu = \frac{\mathrm{B}}{\mathrm{H}}$	$f(\mathrm{B}) = 0{\cdot}8\ \mathrm{H}$
5000	1·6	3000	1·28
9000	4	2250	3·2
10000	5	2000	4
11000	6·5	1692	5·2
12000	8·5	1412	6·8
13000	12	1083	9·6
14000	17	823	13·6
15000	28·5	526	22·8
16000	50	320	40
17000	105	161	84
18000	200	90	160
19000	350	54	280
20000	666	30	532·8

As an example of the application of this method another solution of the exercise, worked on p. 205, is added.

Worked Example. An iron ring 5 sq. centimetres cross section and 10 centimetres mean radius has an air gap 5 mms. wide cut across it. The ring is wound uniformly with a number of turns, so that when 5 amperes are sent through the wire the strength of the resulting field in the gap is 10000 C.G.S. lines. Given that the permeability of the iron is 2000 for the excitation used, determine the number of turns.

Let l_i = length of the iron part of the circuit

$$= 2\pi r - {\cdot}5 = 2 \times 3{\cdot}1416 \times 10 - {\cdot}5$$

$$= 62{\cdot}332 \text{ cms.}$$

l_g = length of the air gap

$$= 0{\cdot}5 \text{ cms.}$$

Then if $\mathrm{B}_g = \mathrm{B}_i$ = the induction or flux-density

$$= 10000$$

the number of ampere-turns required for the iron part is

$$CS_i = f(B_i) \times l_i$$
$$= f(10000) \times 62{\cdot}332.$$

Now, by comparing the data given respecting the magnetic properties of the iron with that given in the table on p. 208 it will be observed that the two correspond, and consequently we may use the value of f (B) there given for B = 10000. That is f (10000) is 4 and

$$CS_i = 4 \times 62{\cdot}332 = 249{\cdot}328 \text{ ampere-turns.}$$

Similarly, the ampere-turns required for the air gap are—

$$CS_g = f(B_g) \times l_g$$
$$= f(10000) \times 0{\cdot}5$$

But since the permeability of air is unity, $B_g = H_g$, therefore $H_g = B_g = 10000$,

and $$f(B_g) = 0{\cdot}8\ H_g = 0{\cdot}8 \times 10000$$
$$= 8000$$

and $$CS_g = 8000 \times 0{\cdot}5 = 4000 \text{ ampere-turns.}$$

$$\therefore \quad \text{total excitation in ampere-turns} = CS_i + CS_g$$
$$= 249{\cdot}328 + 4000$$
$$= 4249{\cdot}328$$

But the current is 5 amperes,

$$\therefore \quad \text{number of turns required} = \frac{4249{\cdot}328}{5}$$
$$= 849.$$

So far we have assumed that the introduction of an air gap introduces no disturbing effects on the uniformity of the magnetization; when we remember, however, that there is no insulator of magnetism it becomes obvious that the existence of air gaps and joints in a magnetic circuit or composite magnetic circuit, as we find them for instance in dynamo-machinery, must be conducive to leakage and be accompanied by *leakage lines of force*, which form *stray magnetic fields*. It thus follows that we must allow for this leakage when calculating the number of ampere-turns required for the field magnet coils of a generator, and that the general expression we have given must be modified accordingly. To simplify the determination of the leakage factor we shall neglect the leakage at the joints and consider that which occurs between the pole pieces of a dynamo. Since there is a tendency for lines of force to leak between two points where there is a difference of magnetic potential, let

Ω = difference of magnetic potential between the pole pieces.
M_1 = permeance of armature and air gaps in series.
M_2 = permeance of the stray field,

and the two branches of the magnetic circuit, i. e. armature path and the stray field, are in parallel, therefore

$$\Omega = N \times \frac{1}{M_1} = N_L \times \frac{1}{M_2}$$

and
$$N_L = N \times \frac{M_2}{M_1}$$

where N = magnetic flux through the armature,
and N_L = leakage flux, or lines which do not pass through the armature,

and if N_m = total flux of lines through the iron of the field magnets

$$N_m = N + N_L = N\left(1 + \frac{M_2}{M_1}\right)$$
$$= N\left(\frac{M_1 + M_2}{M_1}\right) = N \times \nu$$

where $\nu = \frac{M_1 + M_2}{M_1}$ = a factor always greater than unity and $\frac{N_m}{N} = \nu$ = coefficient of magnetic leakage
= ratio of total flux to useful flux.

It is thus obvious that in the iron parts of the magnetic circuit through which N_m lines pass, the saturation will be greater than in the air gaps. But permeability decreases rapidly as the saturation increases, or, in other words, the reluctance of the complete circuit increases, consequently additional excitation or magnetizing influence is required to give a flux, N, in the armature. Hopkinson was the first to recognize this fact, and in his classical paper on 'Dynamo-Electric Machinery,' he introduced this term, coefficient of leakage. Esson, who has made a large number of tests on different machines, finds that the value of ν varies from 1·3 to 1·5 for modern machines.

A little consideration will show that the value of ν depends upon the type of machine, shape of the magnetic system, the presence and position of adjacent masses of iron in the bed-plate, &c., and upon the amount of current taken from the armature since the demagnetizing effect depends upon the armature current. The leakage co-efficient, ν, is therefore only constant for a certain set of conditions. The following paragraph, taken from Hopkinson's paper, 'Dynamo-Electric Machinery,' will explain how the value of ν is determined experimentally:—

'Around the middle of one of the magnet limbs a single coil of wire was taken, forming one complete revolution, and its ends connected to a Thomson's mirror galvanometer rendered fairly ballistic. If the circuit of the field magnets, whilst the exciting current is passing, be suddenly short-circuited, the elongation of the

galvanometer is a measure of the total induction within the core of the limbs, neglecting the residual magnetism. If the short circuit be suddenly removed, so that the current again passes round the field magnets, the elongation of the galvanometer will be equal in magnitude and opposite in direction.

The readings taken were

Zero	71 left	
Deflection	332 ,,	magnets made
,,	196 right;	magnets short-circuited
,, to right =	267	
∴ ,, left =	261	
Mean deflection =	264.	

To determine the induction through the armature, the leads to the ballistic galvanometer were soldered to consecutive bars of the commutator, connected to that convolution of the armature which lay in the plane of commutation.

The readings taken were

Zero	23 left	
Deflection	223 ,,	magnets made
,,	176 right;	magnets short-circuited
,,	178 right;	magnets short-circuited
Mean deflection	200.	

It thus appears that out of 264 lines of force passing through the cores of the magnet limbs at their centre, 200 go through the cores of the armature, whence $\nu = 1{\cdot}32$. The magnetizing current round the fields during these experiments was 5 amperes.

In addition to the actual leakage lines to which we have referred, there is a leakage or spreading out of the lines of force between the pole pieces and the armature core, which forms a fringe on all sides of the air gap. The effective area of the air gap is thus greater than its mean area subtended by the polar surfaces, and given by the angle of embrace of the poles; the margin to be added is given very approximately by taking on each side a length equal to four-fifths of the iron to iron space. The method of making this correction will be apparent from the fully-worked example which is given below.

We may now give the modified form of the general expression for the magnetic circuit of a dynamo, and include the correction factors rendered necessary by reason of magnetic leakage. Using the usual symbols we have, for the number of ampere-turns

$$CS = f(B_a) \times l + 0{\cdot}8\ B_g \times 2\, l_g + f(\nu B_p) \times l_p + f(\nu B_m)\, l_m + f(\nu B_y) \times l_y$$

and, as this expression indicates, it is advisable to determine the number of ampere-turns required for each separate part, the sum of which gives the total number of ampere-turns required for the machine, if the demagnetizing effect of the armature current be neglected. This is the method which will be adopted in the following worked out example.

Worked Example. It is required to determine the number of ampere-turns required to produce a magnetic flux through the armature of a bipolar dynamo = 10826000 lines. The armature is of the drum type, and is built

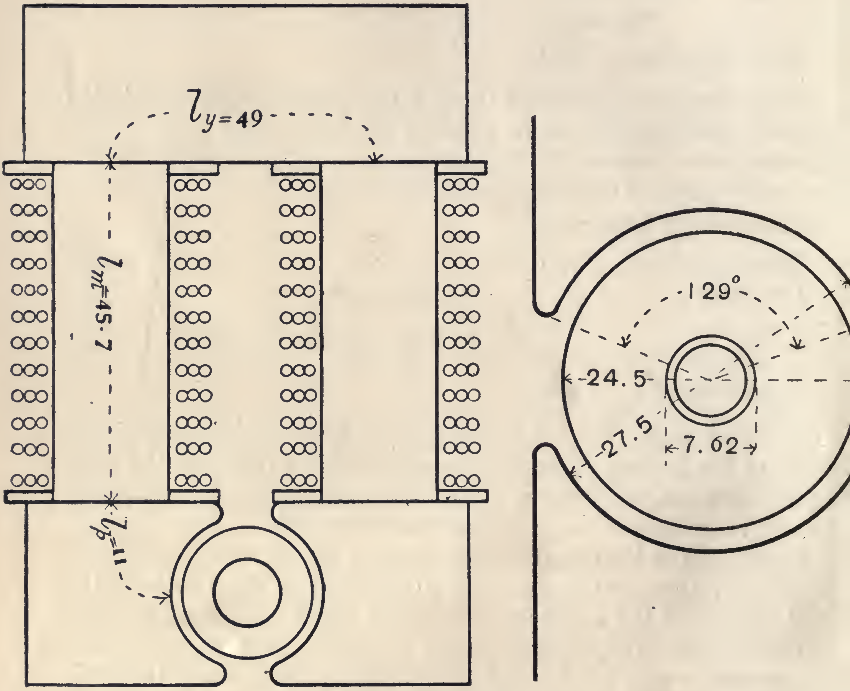


Fig. 39.

up of 1000 discs of soft iron 0·474 mm. thick, and the total thickness of paper, insulating the discs, is 3·4 cms. The pole-pieces, field-magnets, and yoke are made up of wrought iron, and the areas of cross section are :—pole-pieces, $A_p = 1230$ sq. cms.; magnet limbs, $A_m = 980$ sq. cms.; yoke, $A_y = 1120$ sq. cms. The other dimensions are given in Fig. 39. The coefficient of leakage is 1·32, and the angular breadth of polar face or angle of embrace is 129°.

The necessary portion of the magnetization curve for wrought iron is given in Fig. 41, p. 214.

(*a*) *Armature core.*

Area of cross-section of armature (Fig. 40 *a*) is

$$A_a = (24{\cdot}5 - 7{\cdot}62)\ (1000 \times {\cdot}474 \times {\cdot}1)$$
$$= 800 \text{ sq. cms.}$$

Note.—·474 mm. = (·474 × ·1) cm.

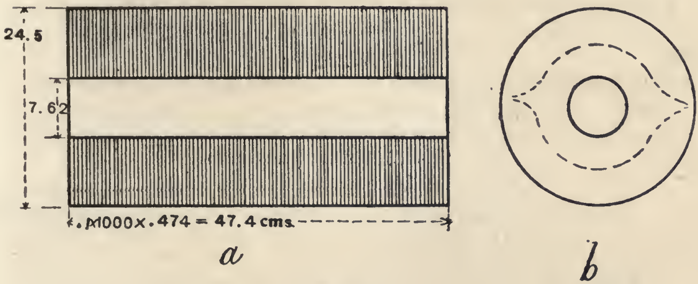


Fig. 40.

The length of the path of the lines of force through the armature may be taken as that given by the dotted line in Fig. 40 *b*.

$$\therefore \quad l_a = \frac{24{\cdot}5 + 7{\cdot}62}{2} \times \frac{\pi}{2} = 25 \text{ cms.}$$

and

$$B_a = \frac{N}{A}$$

$$\therefore \quad B_a = \frac{10826000}{800} = 13532{\cdot}5$$

and the number of ampere-turns required for the armature is

$$\Omega_a = CS_a = f\,(B_a) \times l_a = f\,(13532{\cdot}5) \times 25,$$

but from Fig. 41

$$f\,(13532{\cdot}5) = 11{\cdot}6$$

$$\therefore \quad \Omega_a = CS_a = 11{\cdot}6 \times 25 = 290.$$

(*b*) *Air gaps.*

The length of the air gap on each side is

$$l_g = \tfrac{1}{2}\,(27{\cdot}5 - 24{\cdot}5) = 1{\cdot}5 \text{ cms.}$$

To get the effective area of the air gap proceed as follows :—

The curved length subtending the polar angle (129°) at the mean radius of the air space is

$$= \pi\ (24{\cdot}5 + 1{\cdot}5) \times \frac{\text{polar angle}}{360^\circ}$$

$$= 3{\cdot}1416 \times 26 \times \frac{129^\circ}{360^\circ} = 29{\cdot}27 \text{ cms.}$$

and the gross length of the armature

$$= (1000 \times {\cdot}474 \times {\cdot}1 + 3{\cdot}4) = 50{\cdot}8 \text{ cms.}$$

To each of these lengths we have to add at each end, as explained above, $\frac{4}{5}$ths of the iron to iron space to obtain the margin which has to be allowed

on account of the fringe of lines of force. Therefore the effective area of the air gap is

$$A_g = (29{\cdot}27 + 2 \times \tfrac{4}{5} \times 1{\cdot}5)\ (50{\cdot}8 + 2 \times \tfrac{4}{5} \times 1{\cdot}5)$$
$$= 31{\cdot}67 \times 53{\cdot}2 \text{ sq. cms.}$$
$$= 1685 \text{ sq. cms.}$$

and $$B_g = \frac{10826000}{1685} = 6461$$

And because the permeability of air is unity

$$f(B_g) = 0{\cdot}8 \times H_g$$

and $$H_g = 6461$$

therefore, the number of ampere-turns required for the two air gaps is

$$\Omega_g = CS_g = f(B_g) \times 2\, l_g$$
$$= 0{\cdot}8 \times 6461 \times 2 \times 1{\cdot}5$$
$$= 15506{\cdot}4 \text{ ampere-turns.}$$

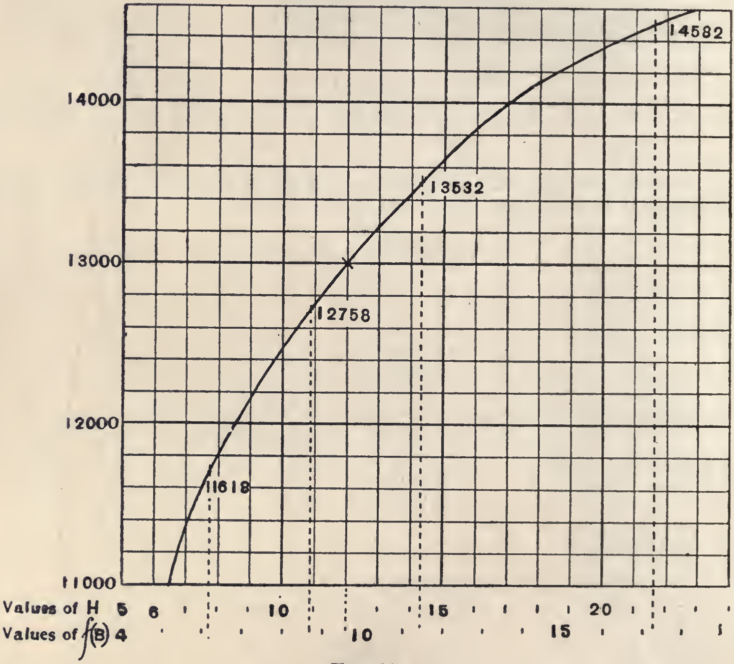


FIG. 41.

(c) *Pole-pieces.* The only point worthy of notice here is the introduction of the co-efficient of leakage, and as we have previously pointed out the total flux in pole-pieces = armature flux $\times\, v$

or $$N_m = N \times v.$$

$$\therefore \quad B_m = \frac{N_m}{A_m}$$
$$= \frac{10826000 \times 1{\cdot}32}{1230}$$
$$= 11618$$

and number of ampere-turns required for the two pole-pieces is

$$\Omega_p = CS_p = f(B_m) \times 2\, l_p$$
$$= f(11618) \times 2 \times 11$$

but $\quad f(11618) = 6{\cdot}1$ (see Fig. 41)

$$\therefore \quad \Omega_p = CS_p = 6{\cdot}1 \times 2 \times 11$$
$$= 134{\cdot}2 \text{ ampere-turns.}$$

(*d*) *Magnet limbs.* Similarly

flux through magnet limbs $= B_m = \dfrac{N_m}{A_m}$

$$= \frac{10826000 \times 1{\cdot}32}{980}$$
$$= 14582$$

and the number of ampere-turns required for the two magnet limbs is

$$\Omega_m = CS_m = f(B_m) \times 2\, l_m$$
$$= f(14582) \times 2 \times 45{\cdot}7$$

but $\quad f(14582) = 17{\cdot}3$ (see Fig. 41)

$$\therefore \quad \Omega_m = CS_m = 17{\cdot}3 \times 91{\cdot}4$$
$$= 1581{\cdot}22 \text{ ampere-turns.}$$

(*e*) *The Yoke.* Similarly

flux through the yoke $= B_y = \dfrac{N_y}{A_y}$

$$= \frac{10826000 \times 1{\cdot}32}{1120}$$
$$= 12759$$

and the number of ampere-turns required for the yoke is

$$\Omega_y = CS_y = f(B_y) \times l_y$$
$$= f(12759) \times 49$$

but $\quad f(12759) = 8.9$ (see Fig. 41)

$$\therefore \quad \Omega_y = CS_y = 8.9 \times 49$$
$$= 436{\cdot}1 \text{ ampere-turns.}$$

Adding together the values of Ω_a, Ω_g, Ω_p, Ω_m, and Ω_y, we get the total magnetizing influence in ampere-turns

For the armature,	$\Omega_a =$	$290{\cdot}$
For the air gaps,	$\Omega_g =$	$15506{\cdot}4$
For the pole-pieces,	$\Omega_p =$	$134{\cdot}2$
For the magnet-limbs,	$\Omega_m =$	$1581{\cdot}22$
For the yoke,	$\Omega_y =$	$436{\cdot}1$
		17948 ampere-turns.

EXERCISES V c.

Magnetic Circuit. II.

(1) An iron ring, of which the mean diameter is 18 cms., is excited by 10 amperes traversing 540 turns of wire. The area of cross section of the iron is 17·5 square centimetres, and the permeability is 800. Determine the magnetomotive force of the coil, the reluctance of the ring, the magnetic induction, and the total magnetic flux.

(2) Determine the flux density and the total magnetic flux if a

saw-cut 5 millimetres wide be made in the above ring, the other factors remaining the same.

(3) What is the magnetic resistance (reluctance) of the above ring with the saw-cut?

(4) How many ampere-turns will be required to produce a flux density of 96000 lines in the above ring with the saw-cut?

(5) A certain dynamo has the following dimensions :—

(*a*) Armature : $l_a = 16$ cms. ; $A_a = 160$ sq. cms. ; $\mu_a = 1000$.
(*g*) Air gaps : l_g (each) $= 0{\cdot}75$ cm. ; $A_g = 800$ sq. cms. ; $\mu_g = 1$.
(*m*) Field magnet cores : $l_m = 70$ cms. ; $A_m = 500$; $\mu_m = 500$.
(*y*) Yoke : $l_y = 30$ cms. ; $A_y = 600$ sq. cms. ; $\mu_y = 400$.

Determine the reluctance of the magnetic circuit.

(6) Determine the magnetic flux through the armature if the excitation consists of 40672 ampere-turns and the coefficient of leakage is 1·4.

(7) A wrought iron ring $7\frac{1}{2}$ cms. mean radius and 1 sq. cm. cross section has a number of turns of wire coiled uniformly upon it, and with a certain current passing along the wire the magnetic field set up is 10000 C.G.S. units, determine the strength of the field when a gap 2·5 mm. wide is cut across the iron. The permeability of the iron may be assumed to be 1000.

(8) If twice as many turns of wire were used how would the current strength have to be altered to give a field of 10000 C.G.S. units in the gap in Exer. 7?

(9) An iron ring 5 sq. cms. cross section and 5 cms. mean radius has a gap 5 mms. wide cut across it. The ring is wound uniformly with a number of turns, so that when 5 amperes are sent through the wire the strength of the resulting field in the gap is 10000 C.G.S. units. Given that the permeability of the iron is 1000, find the number of turns.

(10) Find the magnetic flux in a circuit made up of 25·4 cms. of wrought iron 26 square cms. in area, 25·4 cms. of cast iron, and an air gap ·5 cm. wide, the area being the same throughout the circuit, if 400 ampere-turns constitute the magnetizing force, given the permeability of wrought iron as 1000, and that of cast iron as 800.

(11) An iron ring is made partly of annealed wrought iron and partly of grey cast iron, the length of the latter being three times that of the former. The section is circular, and the inner and outer diameters are 31·825 and 34·081 cms. There are 5 turns of wire per centimetre all round the ring. Determine the flux density when 0·8 and 3 amperes respectively are sent through the coil, the values of the permeability being taken from the table on p. 208.

(12) A ring is made of round wrought iron, and the mean diameter is 40 cms., but the area of cross section is not uniform; two-fifths of the ring being 2 cms. diameter, and the remainder 2·5 cms. diameter. Determine the number of ampere-turns required to produce a flux density of 15000 lines in the thinnest part, the winding being distributed uniformly, μ being taken as 1000.

(13) A dynamo, the dimensions of which are given below, has 2484 turns of wire on the field magnets, determine the total magnetic flux when 2·5 amperes traverse the coils.

	length.	area.	ft.
Field magnets	100 cms.	400 sq. cms.	2500
Air gap	1·5 cms.	800 sq. cms.	
Armature core	20 cms.	200 sq. cms.	2000

(14) Calculate the number of ampere-turns necessary to send 16000 lines per sq. cm. through the armature of a dynamo, in which

internal	diameter	of	armature	core is	8	cms.
external	,,	,,	,,	,,	25	,,
length	,,	,,	,,	,,	50	,,

length of air gap = 1·5 cms.; breadth of polar face = 130°; cross section of magnet cores = 46 cms. × 22 cms.; mean length of lines is 150 cms.; leakage coefficient = 1·4; and the permeability of iron is 800.

Section IV. Fröhlich's Law.

§ 49. Lamont-Fröhlich's Law of the Electromagnet.—By means of Hopkinson's Law, already explained, we are to a certain extent able to deduce the properties of a dynamo machine by considering the relation between the magnetic field and the magnetizing influence. As will be explained in Chapter VII, the electromotive force, E, induced in an armature, is proportional to the magnetic flux linked with the armature coils and to the speed of the machine; we may, therefore, in practice make a further deduction by means of which the relation between the E.M.F. produced in an armature at a given speed, and the exciting current in the field magnet coils, may be determined. It thus follows that we may express the E.M.F. as an empirical function of the magnetizing current with a certain degree of accuracy.

The empirical formula which agrees best with experimental results is that due to Fröhlich, announced in 1881, and which is an extension of Lamont's Law previously proposed. The following shows that Fröhlich's Law may be deduced from information supplied

by the magnetization curves of samples of iron and steel employed in dynamo construction, and that the law conforms within certain limits to modern practice. In Fig. 42 the magnetization curve, I, for a sample of steel forging for dynamo magnets has been constructed

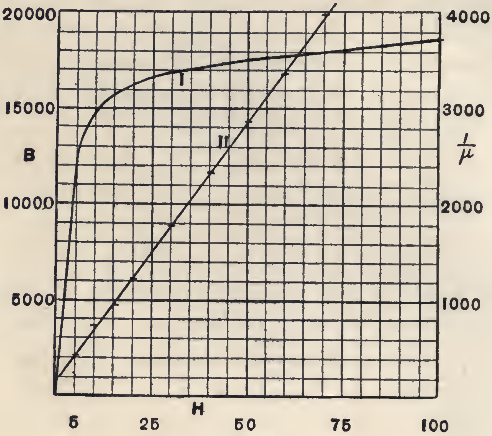


Fig. 42.

from the following data, given by Professor Ewing in his paper on 'The Magnetic Testing of Iron and Steel,' read before the Institution of Civil Engineers, May 19, 1896:—

Magnetizing Force. H	Induction. B	Permeability. μ	Reluctivity. $\frac{1}{\mu}$ or K
5	12300	2460	406
10	14920	1492	670
15	15800	1054	949
20	16280	814	1228
30	16810	560	1785
40	17190	429	2331
50	17500	350	2857
60	17750	296	3375
70	17970	257	3891
80	18180	225	4444
90	18390	204	4902
100	18600	186	5376

The figures in the 4th column give the reluctivity, $\frac{1}{\mu}$, or reluctance of a centimetre cube; this is obviously always less than unity,

and the numbers given in this column are expressed in terms of a unit which is the millionth part of the reluctivity of air. Curve II represents the reluctivity of the sample of steel forging for dynamo magnets, and as will be observed the reluctivity follows a straight line law. There is thus the linear relation

$$\frac{1}{\mu} = a^1 + b^1 \,\mathsf{H} = \text{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I})$$

existing between reluctivity and magnetizing force, and since

$$\mathsf{H} = \frac{4\,\pi}{10\,l} \times \text{CS and } \mathsf{B} = \mu\,\mathsf{H}$$

we get the following relation between induction B and the number of ampere-turns CS. From the linear relation

$$\frac{1}{\mu} = a^1 + b^1 \,\mathsf{H}$$

we have

$$\mu = \frac{1}{a^1 + b^1 \,\mathsf{H}}$$

and

$$\mathsf{B} = \mu\,\mathsf{H} = \frac{\frac{4\,\pi}{10\,l}\,\text{CS}}{a^1 + b^1 \times \frac{4\,\pi}{10\,l}\,\text{CS}}$$

$$= \frac{a\,\text{CS}}{1 + b\,\text{CS}} = \frac{\text{C}}{\alpha + \beta\,\text{C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II})$$

where a, b, α, and β are constants such that

$$a = \frac{4\,\pi}{10\,l\,a^1} \quad \text{and} \quad b = \frac{4\,\pi\,b^1}{10\,l\,a^1}$$

$$\alpha = \frac{a^1}{\frac{4\pi\,\text{S}}{10\,l}} \quad \text{and} \quad \beta = \frac{b}{a} = b^1.$$

The expression in equation I represents Frohlich's Law in a simple form, and it is interesting to notice that Messrs. Houston and Kennelly give in their 'Electro-Dynamic Machinery' a short table of values of a^1 and b^1 for the relation $\text{K} = a^1 + b^1\,\mathsf{H}$ for different kinds of iron and steel used for the cores of field magnets. They are as follows:—

For ordinary dynamo cast iron	$\text{K} = 0{\cdot}0026 + 0{\cdot}000093\,\mathsf{H}$
For dynamo wrought iron	$\text{K} = 0{\cdot}0004 + 0{\cdot}000057\,\mathsf{H}$
For soft iron (Stoletow)	$\text{K} = 0{\cdot}0002 + 0{\cdot}000056\,\mathsf{H}$
For cast iron	$\text{K} = 0{\cdot}0010 + 0{\cdot}000129\,\mathsf{H}$
For Norway iron	$\text{K} = 0{\cdot}0001 + 0{\cdot}000059\,\mathsf{H}$
For steel	$\text{K} = 0{\cdot}00045 + 0{\cdot}000051\,\mathsf{H}$

These values give good results in calculations relating to closed magnetic circuits, consisting entirely of iron as in the case of an iron ring, and for examples dealing with compound magnetic circuits containing an air gap fairly good results may be obtained if the fundamental relationship $\mathrm{K} = \frac{1}{\mu} = a^1 + b^1\,\mathsf{H}$ be modified so as to introduce the flux density B instead of the magnetizing force, H.

Since $\mathsf{B} = \mu\,\mathsf{H}$ and $\mu = \frac{1}{\mathrm{K}}$

we have $\mathsf{H} = \mathrm{K}\,\mathsf{B}$

$$\therefore \quad \mathrm{K} = a^1 + b^1\,\mathrm{K}\,\mathsf{B}$$

and
$$\mathrm{K} = \frac{a^1}{1 - b^1\,\mathsf{B}}$$

As an example of these formulae we may refer to the soft iron ring considered previously on p. 205, and it will be found that the results agree fairly well with those already obtained.

Worked Examples. (1) A soft iron ring, 5 sq. cms. in cross-sectional area and of 10 cms. mean radius, is wound with 845 turns of wire carrying a current of $\frac{50}{169}$ ampere. Determine the reluctance of the magnetic circuit, the total magnetic flux and the permeability of the iron.

Since $\mathsf{H} = 0{\cdot}4\,\pi\frac{\mathrm{CS}}{l}$

the magnetizing force is
$$\mathsf{H} = 0{\cdot}4\,\pi \times \frac{50}{169} \times \frac{845}{20\,\pi}$$
$$= 5 \text{ units.}$$

By applying the values of a^1 and b^1 for soft iron given on p. 219 we have for the reluctivity

$$\mathrm{K} = 0{\cdot}0002 + 0{\cdot}000056\,\mathsf{H}$$
$$= 0{\cdot}0002 + 0{\cdot}000056 \times 55 \times 5$$
$$= 0{\cdot}00048$$

$$\therefore \quad \text{reluctance } \mathcal{R} = \mathrm{K} \times \frac{l}{\mathrm{A}} = 0{\cdot}00048 \times \frac{20 \times \pi}{5}$$
$$= 0{\cdot}006031872$$

and permeability
$$= \mu = \frac{1}{\mathrm{K}} = \frac{1}{0{\cdot}00048} = 2033{\cdot}3.$$

Again, since magnetic flux $= \frac{\text{M.M.F.}}{\text{reluctance}}$

$$\mathsf{N} = \frac{1{\cdot}257\ \mathrm{CS}}{0{\cdot}006031872}$$
$$= 1{\cdot}257 \times \frac{50}{169} \times \frac{845}{0{\cdot}00603}$$
$$= 52114 \quad .$$

(2) Determine the current required to produce a flux density of 10000 lines if a saw-cut 5 mms. wide be made in the above ring.

The reluctance of the air gap is

$$\mathcal{R}_g = \frac{\cdot 5}{5} = 0{\cdot}1$$

The reluctance of the iron is

$$\mathcal{R}_i = \mathrm{K}_i \times \frac{l_i}{\mathrm{A}}$$

and

$$\mathrm{K}_i = \frac{a^1}{1 - b^1 \mathrm{B}}$$

$$= \frac{0{\cdot}0002}{1 - 0{\cdot}000056 \times 10000} = \frac{0{\cdot}0002}{0{\cdot}64}$$

$$= 0{\cdot}0003125$$

$$\therefore \quad \mathcal{R}_i = 0{\cdot}0003125 \times \frac{20\pi - \cdot 5}{5}$$

$$= 0{\cdot}00389575$$

and

$$\mathcal{R} = \mathcal{R}_g + \mathcal{R}_i = 0{\cdot}1 + {\cdot}00389575$$

$$= 0{\cdot}100389575$$

and since

$$\mathrm{N} = \frac{\text{M.M.F.}}{\mathcal{R}}$$

$$\therefore \quad 50000 \times 0{\cdot}100389575 = 1{\cdot}257 \times \mathrm{C} \times 845$$

$$\therefore \quad \mathrm{C} = \frac{5019478{\cdot}75}{1062165}$$

$$= 4{\cdot}7 \text{ amperes}$$

In the previous solution 5 amperes was the value of the current.

In Fröhlich's researches upon the law of the electromagnet he assumed that E, the E.M.F. in the armature of a dynamo, was proportional to the speed n and to a quantity M, which he called the *effective magnetism*, and which more recently has been termed *induction factor* by Professor Carus-Wilson (see Chapter IX). M is proportional to both the effective area of the armature coils and to the intensity of the magnetic field. We may therefore write

$$\mathrm{E} = n\,\mathrm{M}.$$

but, by Ohm's Law

$$\mathrm{E} = \mathrm{CR}$$

$$\therefore \quad \frac{\mathrm{C}}{\mathrm{M}} = \frac{n}{\mathrm{R}}$$

From this relationship it is obvious that M is a function of C, but not primarily of n or R, whilst C is a function of $\frac{n}{\mathrm{R}}$. By observing the value of C for different values of n and R, Fröhlich endeavoured to determine experimentally this function $\frac{n}{\mathrm{R}} = f(\mathrm{C})$, and by taking the values of $\frac{n}{\mathrm{R}}$ as abscissae, and the corresponding values of C as ordinates, he plotted his results as a curve, which was found to be similar to the curve AB, shown in Fig. 43. This curve he termed the 'current curve,' and, as shown, the current

curve is a straight line not passing through the origin O. We may also observe here that it is almost impossible to obtain observations for the curved portion BC, since it corresponds to values of n and R, which renders the excitation of the field magnets of a series machine practically impossible. We may therefore conclude that the straight portion AB corresponds to values corresponding to the practical working conditions of the dynamo. The linear equation to the current in Fig. 43 may be written

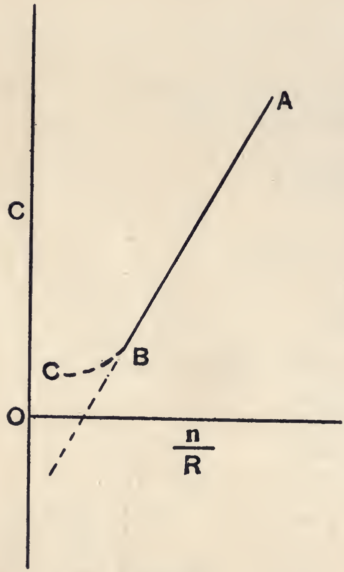


FIG. 43.

$$\frac{n}{R} = a + \beta C$$

where a and β are constants.

$$\therefore \quad \frac{C}{M} = a + \beta C$$

and
$$M = \frac{C}{a + \beta C}$$

which corresponds to one form of the Fröhlich formula (II) previously obtained. Professor S. P. Thompson has largely used Fröhlich's ideas in his treatment of the magnetic principles of a dynamo, and very early introduced the formula

$$H = \frac{G\,k\,CS}{1 + \sigma CS},$$

in which H = resulting average intensity of the magnetic field,
k = initial value of the magnetic permeability,
G = a coefficient depending upon the size and form of the core,
σ = saturation coefficient, which is such that its reciprocal $\frac{1}{\sigma}$ = that number of ampere-turns which will give to the magnet exactly *half* its *maximum magnetism*. Thompson's equation may be written

$$H = \frac{G\,k}{\sigma} \times \frac{CS}{\frac{1}{\sigma} + CS}$$

$$= Y \frac{CS}{(CS)^1 + CS}$$

in which $(CS)^1$ = ampere-turns which will give half saturation to the magnetism. More recently he has given the equation in a new form, i. e.

$$N = N^1 \frac{CS}{(CS)^1 + CS}$$

in which N = magnetic flux
and N^1 = maximum magnetic flux which occurs at complete saturation.

Professor Thompson terms the ampere-turn $(CS)^1$ the *diacritical* number, and the current C^1 the *diacritical* current.

Obviously, if S is known, we may cancel it out and write

$$N = N^1 \frac{C}{C^1 + C}$$

C^1 being the diacritical or half-saturating current. If r be the resistance of the magnetizing coil, then since $Cr = e$, the difference of potential required to send the current through coil of resistance, r, the last equation becomes

$$N = N^1 \frac{e}{e^1 + e}$$

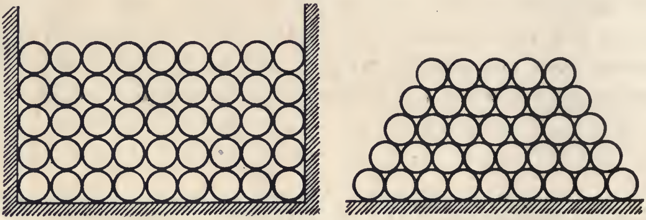
The two latter forms of the equation are very simple, and they are approximately true for electromagnets excited by a single current. Two observations made on any electromagnet will enable us to determine the two constants N^1 and C^1, and Professor Thompson has shown how the above formulae may be applied to calculations of shunt-wound machines as well as to series-wound machines.

CHAPTER VI

COIL-WINDING DESIGN

Section I. Coils for Electro-magnets.

§ 50. Simple Formulae. When the number of the ampere-turns required to produce a given magnetic flux for a magnetic circuit has been determined, there are many interesting problems respecting the disposal of the coil windings, the determination of the length and diameter of the turns to offer a given resistance which may be wound on a bobbin of given dimensions, to be solved. To simplify the general solution we propose to treat each phase of the problem separately, and to proceed step by step in the simplest manner possible. In the first place we must distinguish between two species of winding: (1) the *rectangular winding*, and (2) the *conical winding*, as shown in Fig. 44.



I. FIG. 44. II.

The rectangular winding is such that the wires of one layer are directly superimposed on those of the layer below so as not to lie in the hollows of the lower layer, the section of the whole forming a rectangle (Fig. 44, I). In the conical winding the wires of one layer lie in the grooves formed by the lower layer, the section of the whole forming a triangle or part of a triangle (Fig. 44, II).

Rectangular Winding. Let Fig. 45 represent the bobbin of an electro-magnet of given dimensions which is filled with cylindrical wire so as to form a rectangular winding, and let

d = the diameter of the insulated wire in inches.
D = the internal diameter of the bobbin in inches.
D_1 = the external diameter of the bobbin in inches.

D_2 = the mean diameter of the coil in inches.
b = the length of bobbin or distance between the cheeks in inches.
l_m = the mean length of a turn in inches.
L = the total length of wire in the coil in inches.
h = the thickness of the coil in inches.

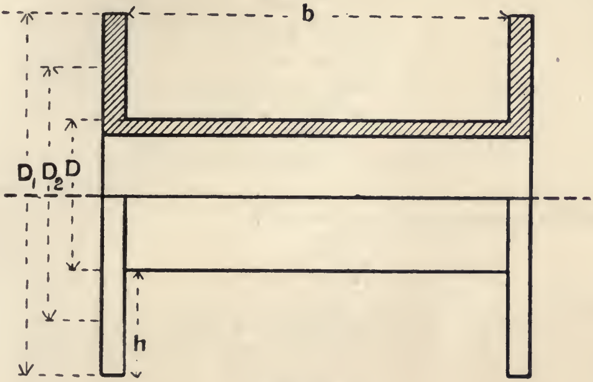


Fig. 45.

It may be mentioned here that it is somewhat difficult to determine exactly the diameter of a wire insulated with a silk or cotton covering by means of a micrometer or other gauge, and the method which gives results corresponding to the practical winding of bobbins is to wind very carefully a close coil, say 10 inches in length, on a glass rod, and so obtain the diameter of the insulated wire from the number occupying the 10 inches. From the geometry of the coil the following relationships are obvious, but the resistance of the coils is left out of consideration for the present.

The number of turns, m, per layer $= \frac{b}{d}$,

and the number of layers $n = \frac{h}{d}$,

$\therefore$ the total number of turns, S, on the bobbin is given by the product of n and m,

and $$n \times m = S = \frac{bh}{d^2}.$$

From this result the diameter of a wire required to fill a bobbin of given dimensions with S turns of wire may readily be determined, thus

$$d = \sqrt{\frac{bh}{S}}.$$

Worked Examples. (1) How many turns of wire are there on a bobbin, if the coil is 8 inches long, 1·6 inches thick, and the diameter of the wire (insulated) is 40 mils?

In this case $d = 0{\cdot}04$ inch,

$$\therefore \quad m = \frac{8}{\cdot 04} = 200$$

and

$$n = \frac{1{\cdot}6}{\cdot 04} = 40,$$

$$\therefore \quad S = mn = \frac{bh}{d^2}$$

$$= \frac{8 \times 1{\cdot}6}{(\cdot 04)^2} = \frac{12{\cdot}8}{\cdot 0016}$$

$$= 8000 \text{ turns.}$$

(2) What must be the diameter of the wire so that 5120 turns exactly occupy the same space as the above coil?

$$d = \sqrt{\frac{bh}{S}}$$

$$= \sqrt{\frac{8 \times 1{\cdot}6}{5120}} = \sqrt{\frac{1}{400}}$$

$$= \frac{1}{20} = 0{\cdot}05 \text{ inch}$$

$$= 50 \text{ mils.}$$

To find L, the total length of wire which may be wound on a bobbin of given dimensions when the diameter d of the wire is given, we must remember that the diameters of the layers increase as the number of layers increase, and for this reason it is advisable to get the value of the mean diameter D_2 of the coil, which is obviously

$$D_2 = \tfrac{1}{2}(D + D_1),$$

$\therefore$ the length of a mean turn is $l_m = \pi D_2 = \frac{\pi}{2}(D + D_1)$.

And since there are S or mn turns in the coil, the total length of wire, L, is

$$L = \text{mean length} \times \text{number of turns}$$

$$= l_m \times S$$

$$= \frac{\pi}{2}(D + D_1) \times mn$$

$$= \frac{\pi}{2}(D + D_1) \times \frac{b}{d} \times \frac{h}{d}$$

$$= \frac{\pi bh(D + D_1)}{2d^2} \text{ inches}$$

$$= \cdot 131 \frac{bh(D + D_1)}{d^2} \text{ feet.}$$

But $$h = \tfrac{1}{2}(D_1 - D),$$

$$\therefore \quad L = \frac{\pi b (D_1 - D)(D_1 + D)}{4d^2}$$

$$= \frac{\pi b (D_1^2 - D^2)}{4d^2} \text{ inches}$$

$$= \frac{\cdot 06545\, b (D_1^2 - D^2)}{d^2} \text{ feet.}$$

If R = internal radius of coil

and R_1 = external radius of coil,

then $$L = \frac{\pi b (R_1^2 - R^2)}{d^2} \text{ inches}$$

$$= \frac{\cdot 262\, b (R_1^2 - R^2)}{d^2} \text{ feet.}$$

Worked Example. Determine the length of wire used to fill the bobbin in example (2), page 226, if $D = 4$ inches and $D_1 = 7{\cdot}2$ inches.

In this case the length of a mean turn is

$$l_m = \frac{\pi}{2}(D + D_1)$$

$$= \frac{3{\cdot}1416}{2}(7{\cdot}2 + 4) = 1{\cdot}5708 \times 11{\cdot}2$$

$$= 17{\cdot}59296 \text{ inches};$$

$$\therefore \quad L = l_m \times S = l_m \times \frac{b}{d} \times \frac{h}{d}$$

$$= 17{\cdot}59296 \times \frac{8}{\cdot 05} \times \frac{1{\cdot}6}{\cdot 05}$$

$$= 17{\cdot}59296 \times 5120$$

$$= 90076 \text{ inches}$$

$$= 7506{\cdot}33 \text{ feet.}$$

The space occupied by a given length and diameter of wire forming a coil—this is not the same as the actual volume of the copper—may be expressed in one or more of the following forms according to the data given :—

The volume, $V_s = Ld^2$ cubic inches

$$= b \times \frac{\pi}{4}\{D_1^2 - D^2\}$$

$$= \cdot 7854\, b (D_1^2 - D^2)$$

but $$D_1 = D + 2h$$

$$\therefore \quad V_s = \frac{\pi b}{4}\{(D + 2h)^2 - D^2\}$$

$$= \pi b \{hD + h^2\}$$

$$= 3{\cdot}1416\, bh (D + h)$$

and since $$d^2 = \frac{V_s}{L}$$

$$\therefore \quad d^2 = 3{\cdot}1416 \times \frac{b}{L} \times h \times (D+h).$$

When it is required to determine the weight, w, of copper wire of given diameter which fills a bobbin of given dimensions, it will be convenient to use a constant, K, for each size of wire, which we may term the *specific length as regards weight*, i.e. the number of feet per pound, and which is given in the accompanying table (page 229).

Since $$K = \text{feet per pound}$$

$$w = \frac{L}{K} \text{ pounds,}$$

and by introducing the values of L as given by the above equations the weight may be readily calculated according to the data given.

In some cases it is desired to rewind a bobbin with a new gauge of wire so that the same weight of wire may be used; thus, if w and b be known, and a wire d_1 inches diameter replaces a wire d inches diameter, we have

$$w = \frac{L}{K} = \frac{L_1}{K_1},$$

$$\therefore \quad L_1 = K_1 w$$

where L_1 = total length of new wire, and K_1 = the specific length (got from table). Also $n_1 = \dfrac{h_1}{d_1}$

but $$d_1^2 = \pi \frac{b}{L_1} h_1 (D + h_1)$$

$$\therefore \quad n_1^2 = \frac{h_1^2 L_1}{\pi b h_1 (D + h_1)} = \frac{h_1 L_1}{\pi b (D + h_1)}$$

and $$n_1 = \sqrt{\frac{h_1 L_1}{\pi b (D + h_1)}}$$

also $$m_1 = \frac{b}{d_1} = \text{number per layer}$$

$$\therefore \quad S_1 = m_1 n_1 = \frac{b}{d_1} \sqrt{\frac{h_1 L_1}{\pi b (D + h_1)}} = \sqrt{\frac{b h_1 L_1}{\pi d_1^2 (D + h_1)}}$$

$$= \frac{L_1}{\pi (D + h_1)} \text{ by substituting value of } d_1^2$$

$$\therefore \quad L_1 = \pi S_1 (D + h_1).$$

The following notes will also be found useful in getting the weight of a coil of copper wire.

Pure copper weighs 555 pounds per cubic foot.

The specific gravity of pure annealed copper is 8·9 at 60° F.

The weight of L feet of copper wire d inches in diameter is

$$w = 3{\cdot}02\, d^2 \mathrm{L} \text{ pounds.}$$

Also $w = \dfrac{d^2}{62{\cdot}57}$ pounds per mile where d = diameter in mils.

Pounds per yard = area in square inches × 11·5625.

Worked Example. Determine the coil space required for 400 turns of wire (insulated) 86 mils in diameter, if the mean length of a turn is 4 feet. Also find the weight of the wire.

Since the mean length $l_m = 4$ feet,

the total length $\mathrm{L} = 400 \times 4 = 1600$ feet,

and
$$\mathrm{V}_s = \mathrm{L}\, d^2 = 1600 \times 12 \times (\cdot 086)^2$$
$$= 142 \text{ cubic inches.}$$

Allowing 16 mils for insulation the diameter of copper is

$$d = \cdot 086 - \cdot 016 = \cdot 070$$

and
$$w = 3{\cdot}02 \times (\cdot 07)^2 \times 1600$$
$$= 23{\cdot}675 \text{ pounds.}$$

Allowing 14 mils for the insulation, the diameter of the copper would be

$$d = \cdot 086 - \cdot 014 = \cdot 072,$$

which is the diameter of a No. 15 S.W.G. wire (see table below); and since K for this wire (see table) is 63·72

$$w = \frac{\mathrm{L}}{\mathrm{K}} = \frac{1600}{63{\cdot}72}$$
$$= 25{\cdot}1 \text{ pounds.}$$

S.W.G.	Diam. inch.	Cross-section sq. inch.	Turns per inch.	Weight per 1000 yds.	K feet per lb.	Resistance per 1000 yds.
22	·028	·00062	23·81	7·12 lbs.	421·2	39·05
21	·032	·00080	21·74	9·3	322·5	29·90
20	·036	·0010	20·00	11·7	254·88	23·62
19	·040	·0012	18·52	14·5	206·46	19·13
18	·048	·0018	16·13	21·	143·34	13·28
17	·056	·0024	14·28	28·48	105·33	9·762
16	·064	·0032	12·83	37·2	80·94	7·478
15	·072	·0040	11·63	47·1	63·72	5·904
14	·080	·0050	10·64	58·1	51·6	4·784
13	·092	·0060	9·44	76·8	39·03	3·617
12	·104	·0085	8·48	98·2	30·54	2·831
11	·116	·0105	7·69	122·2	24·56	2·275
10	·128	·0128	7·04	148·8	20·16	1·868
9	·144	·0163	6·33	188·4	15·95	1·476
8	·160	·0201	5·74	232·5	12·9	1·196
7	·176	·0243	5·26	281·3	10·66	0·985

Conical Winding. In the case of conical winding (see Fig. 44, II, p. 224) the formulae given previously for the rectangular winding require considerable modification in consequence of the different disposition of the wire. If the section of the coil be a cone, or part of a cone, as in Fig. 44, II, it is obvious that the number in any layer differs from that in the next layer by one, so that if there be n layers, and a turns in the bottom layer, the total number of turns will be

$$S = a + (a-1) + (a-2) + \ldots + [a - (n-1)],$$

the number in the top layer being

$$[a - (n-1)] \text{ or } (a + 1 - n).$$

Now S is obviously the sum of a series of numbers in arithmetical progression which may be shown to be

$$\left.\begin{aligned} S &= na + \frac{n(n-1)}{2} \\ &= na + \frac{n(n-1)}{2}c \end{aligned}\right\}$$

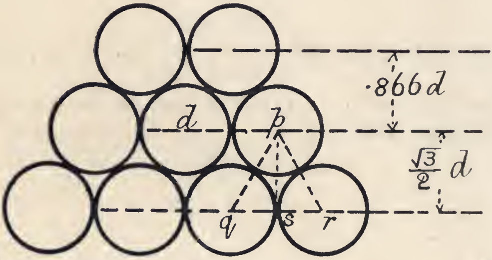


Fig. 46.

if c = common difference between the number of turns in adjacent layers.

In general terms, the number in the top layer is therefore

$$a - (n-1)c.$$

Again, the number of layers for a given thickness of coil, h, is greater than $\frac{h}{d}$. Let Fig. 46 represent a portion of three layers, and as shown, the lines joining the centres of three adjacent wires form an equilateral triangle, as at $p\,q\,r$, the side of the triangle being equal to the diameter of the wire. By geometry it is obvious that the distance between the centres of two adjacent layers is ps, ps being perpendicular to, and bisecting, qr. Therefore $qs = sr = \frac{1}{2}d$, and we have

$$(pq)^2 = (ps)^2 + (qs)^2$$

or
$$d^2 = (ps)^2 + \left(\frac{d}{2}\right)^2 \text{ or } (ps)^2 = d^2 - \frac{d^2}{4}$$

and
$$ps = \sqrt{\frac{3}{4}d^2} = \frac{\sqrt{3}}{2}\,d = \frac{1{\cdot}732}{2}\,d$$

therefore the distance between the layers = $\cdot 866 \times$ diameter of the wire.

From this we get the thickness of a coil of n layers

$$h = \frac{\sqrt{3}}{2}\,d\,(n-1) + d = \left\{\frac{\sqrt{3}}{2}(n-1) + 1\right\} \times d$$

and the number of layers, n, for thickness of coil h is

$$n = \left(\frac{2h}{d} + \sqrt{3} - 2\right)\frac{1}{\sqrt{3}}.$$

It may also be shown that

$$n = (\mathrm{S} + \tfrac{1}{2}) - \sqrt{(\mathrm{S} + \tfrac{1}{2})^2 - 2a}$$

where S = total number of turns in the coil

a = number of turns in the bottom layer.

For approximate work we may often assume that the thickness of the cotton covering is proportional to the thickness of the wire used and which it insulates, in which case it is obvious that the weight of the coil occupying a certain space will be the same for wires of different diameters. We may, however, mention that it is not wise to make this assumption when dealing with very fine wires, since the thickness of the insulating covering bears a greater ratio to the diameter of the wire for wires of small gauge than for wires of large gauge.

Resistance of Coils. The next point to be considered is the relationship between the resistance of the wire forming a coil and its length and diameter, and the number of turns forming the coil. Obviously, when the total length and the diameter of the wire are known, the simplest method of finding the resistance is to refer to a wire-table, giving the number of feet per ohm or the resistance per 1000 yards, as was included in the table given on page 229. For general purposes, however, we may make use of the fundamental formula

$$\mathrm{R} = \frac{\sigma \mathrm{L}}{\mathrm{A}} = \frac{\sigma \mathrm{L}}{\frac{\pi}{4}\delta^2}$$

in which R = resistance of a wire L inches long and of cross-section A square inches.

δ = diameter of bare wire in inches.

σ = resistivity, or resistance of an inch cube of the wire.

For commercial copper—

$$\sigma = 0{\cdot}000000\,6756 \text{ ohm at } 15^\circ \text{ C. or } 60^\circ \text{ F. (per inch cube)}$$
$$= 0{\cdot}00000\,1729 \text{ ohm at } 15^\circ \text{ C. or } 60^\circ \text{ F. (per cm. cube).}$$

For any bobbin, then, on which there are L inches of wire δ inch in diameter, we have

$$R = \frac{L \times r_1}{12} \text{ if } r_1 = \text{resistance per foot}$$

and

$$R = 0{\cdot}000000\,6756 \times \frac{L}{\frac{\pi}{4}\delta^2}$$

$$= \frac{2{\cdot}7024}{10^6} \times \frac{L}{\pi\delta^2} \text{ ohms}$$

and

$$L = \pi\delta^2 R \times \frac{10^6}{2{\cdot}7024}.$$

But we have

$$L = \frac{V_s}{d^2}$$

therefore

$$\frac{V_s}{d^2} = \pi\delta^2 R \times \frac{10^6}{2{\cdot}7024}$$

and

$$R = \frac{V_s}{\pi\delta^2 d^2} \times \frac{2{\cdot}7024}{10^6} \text{ ohms}$$

and since

$$d = \delta + t$$

where t = increase of diameter due to the insulating covering, we may, if t is so small as to be negligible, write

$$R = \frac{V_s}{\pi\delta^4} \times \frac{2{\cdot}7024}{10^6} \text{ ohms.}$$

We see, therefore, that for all coils which fill a given space

(1) *the number of ohms of resistance of the coil is directly proportional to the volume occupied*, and that

(2) *the resistance is inversely proportional to the 4th power of the diameter of the wire used.*

Making the assumption that t is small, we have

$$\delta^4 = \frac{V_s}{\pi R} \times \frac{2{\cdot}7024}{10^6}$$

from which we may obtain the diameter of the wire which will fill a bobbin of given dimensions, and offer a resistance R, that is

$$\delta = \sqrt[4]{\frac{V_s}{\pi R} \times \frac{2{\cdot}7024}{10^6}}$$

$$= 0{\cdot}03045 \sqrt[4]{\frac{V_s}{R}}$$

$$= 0{\cdot}040545\sqrt[4]{\frac{bh(D+h)}{R}}, \text{ since } V_s = \pi bh(D+h).$$

In many cases it will be found advisable to take the size next smaller than the one calculated, so that the wire may be readily wound on the bobbin.

We may also determine the relationship between the resistances and number of turns of different sizes of wire filling a given space on a bobbin. Let

d_1 and d_2 = the diameters of the two wires used.
L_1 ,, L_2 = the lengths ,, ,, ,,
R_1 ,, R_2 = the resistances ,, ,, ,,
S_1 ,, S_2 = the number of turns of the two wires used,

then since $$\frac{R_1}{R_2} = \frac{L_1 d_2^2}{L_2 d_1^2} \text{ and } \frac{L_1}{L_2} = \frac{l_m S_1}{l_m S_2} = \frac{S_1}{S_2}$$

and $$V = L_1 d_1^2 = L_2 d_2^2$$

$$\therefore \quad \frac{L_1}{L_2} = \frac{d_2^2}{d_1^2} = \frac{S_1}{S_2}$$

and by substitution

$$\frac{R_1}{R_2} = \frac{S_1}{S_2} \times \frac{S_1}{S_2} = \frac{S_1^2}{S_2^2}.$$

Or, in words, *the resistance in ohms varies directly as the square of the number of turns in a coil filling a given space.*

This is, of course, on the assumption that d and δ are approximately the same.

Worked Example. It is required to determine the length of No. 16 gauge wire which fills 147·46 cubic inches of coil space, the resistance of the wire being 7·56 ohms.

Since $$d^2 = \frac{V_s}{L}$$

and $$d^2 = \frac{4\sigma L}{\pi R}$$

$$\therefore \quad \frac{4\sigma L}{\pi R} = \frac{V_s}{L} \text{ and } L^2 = \frac{\pi V_s R}{4\sigma}$$

$$\therefore \quad L = \sqrt{\frac{\pi V_s R}{4\sigma}}.$$

And, as will be observed, the right-hand member only contains quantities which are given or known, therefore by substitution we have

$$L = \sqrt{\frac{3{\cdot}1416 \times 147{\cdot}46 \times 7{\cdot}56}{4 \times {\cdot}000000\,6756}}$$
$$= 36000 \text{ inches}$$
$$= 1000 \text{ yards.}$$

EXERCISES VI A.

Coil Winding.

(1) An electromagnet bobbin, 2 inches long, 1 inch external and $\frac{1}{2}$ inch internal diameter, is to be filled with wire 20 mils in diameter; what length of wire will be required? (C. and G.)

(2) What volume is occupied by the wire in No. 1?

(3) Determine the number of turns of wire required to fill a bobbin, the interflange length being $1\frac{5}{8}$ inches, outer diameter 0·5 inch, inner diameter 0·3 inch, if the diameter of the insulated wire is 8 mils.

(4) Determine the total length of wire used in exercise No. 3.

(5) Determine the resistance of the coil in exercise No. 3 if resistance per yard = ·99 ohm.

(6) What will be the outside diameter of the coil if 4000 turns of insulated wire, 5 mils diameter, are wound upon the bobbin in No. 3?

(7) What will be the interflange length required for a coil 400 ohms resistance, if the outer diameter of the coil is 0·75 inch and the inner diameter 0·4 inch, if the diameter of the insulated wire is 9 mils, and each foot of the wire has 0·4 ohm resistance?

(8) Determine the diameter of the insulated wire which just fills the bobbin referred to in exercise No. 7, so that 2500 turns form the coil.

(9) Prove that the total length of wire of diameter (over insulation) d mils which can be wound on a bobbin of interflange length l inches, outer diameter of coil D_1 inches, and inner diameter D inches, is L feet when $L = \frac{65450\, l\, (D_1^2 - D^2)}{d^2}$.

(10) Determine the interflange length of a bobbin required to hold 207·7 feet of wire of diameter (insulated) 8·115 mils, if the outer diameter of the coil is ·6875 inch and the inner diameter ·29 inch.

(11) Determine the outside diameter of a coil which contains 2000 feet of wire of diameter (insulated) 15 mils, if the inner diameter is 0·3 inch and the interflange length 3·5 inches.

(12) If the resistance of a foot of copper wire one mil in diameter is 10·35 ohms, prove that the resistance of L feet of wire, the diameter (bare) of which is δ mils, is $R = \frac{677400\, (D_1^2 - D^2)\, l}{d^2 \delta^2}$, if d, D, and D_1 refer to the same dimensions as in No. 9.

(13) If V is the volume of the winding-space of a bobbin in cubic inches, prove that $R = \frac{862500\, V}{d^2 \delta^2}$.

(14) Two electromagnet bobbins are fully wound with the same weight of wire, the latter being 4 mils gauge in the one case and 2 mils gauge in the other. What will be the relative resistances of the two bobbins? (C. and G.)

(15) Two electromagnets, similar in size, have resistances of 1 and 16 respectively, the weight of wire on both magnets being the same. Compare the diameters of the wires (bare) and the number of turns of wire on the two electromagnets.

(16) An electromagnet is wound to a resistance of 320 ohms, with wire 20 mils in diameter. What diameter would the wire have to be in order that with the same weight of wire on the electromagnet the resistance may be 20 ohms? (C. and G.)

(17) The resistance of the wire on a bobbin fully wound with silk-covered wire 7 mils diameter is found to be 120 ohms. What will be the resistance if the same bobbin be equally wound with 10 mils wire? The thickness of covering to be taken as 1 mil in each case. (C. and G.)

§ 51. **Properties of Electromagnets.** The particular form which must be given to an electromagnet for some specific purpose depends largely upon the magnetic properties of the core and winding of the coil. Thus the design of an electromagnet depends upon the nature of the work it has to perform, and electromagnets employed in telegraph instruments, signalling appliances, chronographs or time-recording apparatus, &c., must be designed for rapid action. It is obvious that the core of such electromagnets should possess residual magnetism and retentiveness to only a slight degree. Consequently the core should be of very pure Swedish iron, well annealed, and possessing a high value of permeability. Hard steel, as is well known, has considerable retentivity, and is altogether unsuitable for the core of such electromagnets, since retentiveness produces a sticking of the armatures in electromagnetic mechanisms. The core should also be short, since it is a general principle of the magnetic circuit 'that in long pieces of iron the mutual action of the various parts tends to maintain the condition of magnetization that they may possess, hence they are less readily demagnetized.' But for rapid action the core of the electromagnet must be quickly magnetized and demagnetized, and 'in short pieces of soft iron the mutual actions are feeble or almost absent, the magnetization is less stable, and disappears almost instantly on the cessation of the magnetizing forces.' This fact may be explained as follows:—The poles of a magnet naturally exert a demagnetizing effect upon the magnet when the poles are not far apart, whilst in the case of long and very thin

cores a considerable amount of magnetism is retained for some time after the removal of the magnetizing forces. It is therefore very important that the cores of electromagnets for rapid action should be short. Another principle of the magnetic circuit which plays an important part in the design of these electromagnets is that a tendency exists for the magnetic circuit to shorten or close itself by attracting the armature of the electromagnet when the coils are energized. And electromagnets which have their poles perfectly bridged by an armature are found to possess some retentiveness when the magnetizing force is removed, even if the core be of soft iron, and the armature be not detached. The residual magnetism, however, disappears immediately the armature is separated from the core. Demagnetization of the core is assisted and hastened by the existence of an air-gap in the magnetic circuit, and the longer the air-path and the shorter the iron-path of a magnetic circuit, the more readily does the iron become demagnetized on the cessation of the current. For rapid action of an armature it is therefore usual to prevent the armature making good contact with the polar faces of the core of the electromagnet by interposing paper or a brass stud, for in such cases free poles always exist, and demagnetization is readily effected. If the exciting current is small, then the number of turns and length of wire must be great to obtain the requisite magnetizing force, and it is for this reason that long cores are sometimes used to provide the space for the winding, but to secure certainty of rapid action it is better to have a short core and to heap up the wire as near as possible to the polar extremities.

Rapidity of action is also accomplished by making the movable parts light and of small inertia, and by placing the armature in a biased position. Rapid action also depends upon another factor, inductance, or self-induction of the circuit, which is a property of electromagnetic coils influencing the growth or cessation of the current more than the actual resistance of the wire, when the circuit is made or broken. Now the inductance of a coil is directly proportional to the square of the number of turns forming the winding, and directly as the permeability of the magnetic circuit. One property of an inductive circuit is to prevent the current rising to its full value when the circuit is closed, as if the circuit possessed 'electric inertia,' and the interval of time which elapses from closing an inductive circuit to the instant that it attains 0·634 of its maximum or steady value is known as the *time-constant* of the coil. The value of this coefficient (time-constant) is given by the ratio of the inductance to the resistance. Now, if the coil consists of two parts (the turns being equal in number), then they may be connected

either 'in series' or 'in parallel'; and from what we have already stated, the time-constant of the coil, when the two parts are connected in parallel, is one-quarter of that which it is when the two parts are connected in series. Consequently advantage is often taken of this fact to diminish the value of the time-constant of the winding, and so improve the rapidity of action.

Rapidity and sensitiveness of action is often obtained by employing polarized mechanisms, in which a permanent steel magnet is introduced to give permanent magnetism to the movable armature which is intended to move between the poles of the electromagnet. The attractive force of the permanent magnet is just balanced by a spring, and the range of motion is restricted by means of studs, so that even small currents will increase the field strength, and so overcome the balance, and the armature moves readily.

Section II. Field Magnets for Dynamos, &c.

§ 52. Windings for Field Magnets. Assuming a given voltage, E_m, at the terminals of the field magnet coils, and a given magnetizing force, it is required to determine the size of the wire to be used.

If we denote the resistance of wire, under the normal working conditions, by R_m, then the current traversing the coils is given by

$$C = \frac{E_m}{R_m}$$

and if L, in feet, be the total length of wire used, we have

$$CL = L \times \frac{E_m}{R_m}$$

which is a useful modification of a quantity usually given in tables of copper wire. As is well known, it is often convenient to know the number of feet per ohm for the different gauges of wire, and this quantity, i.e. feet per ohm, we may term the *specific length as regards resistance*, and is to be found in most tables. Now, if a pressure of one volt be applied to the ends of a length, L, of wire, the resistance of which is one ohm, one ampere will traverse the wire, and we shall have L ampere-feet corresponding to one-volt pressure, and it is obvious that there will be a definite number of ampere-feet corresponding to one-volt pressure for each size of wire, whatever the length, provided the temperature remains the same, i.e. the ampere-feet per volt for any wire is the same as the number of feet per ohm. Also, the product of the voltage used and the number of ampere-feet per volt of the wire gives the total number

of ampere-feet for that wire for the voltage supplied, and this is the quantity given in the above equation.

Worked Example. The ampere-feet per volt of a No. 20 S.W.G. copper is 129, determine the number of ampere-feet which are available when 100 feet of this wire are used and a pressure of 2 volts is maintained at its ends.

For a pressure of 1 volt the current traversing 100 feet of this wire will be $\frac{129}{100}$ or 1·29 amperes, since the resistance of 100 feet is $\frac{100}{129}$ ohm,

and
$$\text{C for 2 volts} = \frac{2}{100/129} = 2{\cdot}58 \text{ amperes}$$
$$\therefore \quad \text{C L} = 2{\cdot}58 \times 100 = 258 = 2 \times 129$$

From the accompanying table it is an easy matter to obtain the right size of wire to use for a given terminal voltage if the number of ampere-turns and the mean length of a turn be known. Thus

$$\text{total number of ampere-feet} = \text{ampere-turns} \times \text{mean length} = (\text{C S}) \times l_m$$
$$\therefore \quad \text{ampere-feet per volt} = \frac{\text{C S}\, l_m}{\text{E}_m}$$

and reference to the table gives the nearest size of wire.

S.W.G.		Feet per ohm or ampere-feet per volt.	S.W.G.		Feet per ohm or ampere-feet per volt.
22	...	76·8	14	...	627·5
21	...	100·3	13	...	829·4
20	...	127·0	12	...	1059·7
19	...	155·8	11	...	1318·7
18	...	225·9	10	...	1606 0
17	...	307·3	9	...	2032·5
16	...	401·1	8	...	2510·4
15	...	500 8	7	...	3036 0

Again, $L = S \times l_m$, and r_1 = resistance per unit length (1 inch) at a temperature of 60° F., so that the resistance of S turns, or L inches, of wire at that temperature is

$$R = S \times l_m \times r_1 = L r_1.$$

But the passage of electricity through a wire is accompanied by the generation of heat and a general rise of temperature, and this goes on until the rate of dissipation of heat by radiation, &c., is equal to the rate of generation, at which point the normal working temperature is reached. The resistance at this stage is

$$R_m = kR = kSl_m r_1$$

where k is a coefficient, the magnitude of which depends on many things, i. e. temperature of surrounding atmosphere, rise of temperature, amount of cooling surface, and the depth of the windings. To this we shall refer later.

Since $$C = \frac{E_m}{R_m}, \quad \therefore \quad CS = \frac{SE_m}{R_m}$$
$$= \frac{SE_m}{kSl_m r_1}$$
$$\therefore \quad r_1 = \frac{E_m}{kl_m CS}.$$

From this relationship the diameter is easily determined, since the resistance of a cubic inch of copper is 0·6756 microhm at 60° F., therefore if A = area of cross-section

$$r_1 = \frac{(\cdot 6756 \times 10^{-6}) \times 1 \text{ inch}}{A}$$
$$\therefore \quad A = \frac{(\cdot 6756 \times 10^{-6}) \times k \times l_m \times CS}{E_m}$$
$$= \frac{\pi}{4}\delta^2$$
$$\therefore \quad \delta = \sqrt{\frac{4 \times \cdot 6756 \times 10^{-6} \times CSl_m k}{\pi E_m}}$$
$$\left.\begin{aligned} &= \sqrt{\frac{\cdot 000000\,8602\, kl_m CS}{E_m}} \\ &= \cdot 0009275 \sqrt{\frac{kl_m CS}{E_m}} \end{aligned}\right\}$$

As this result indicates, there is, for a given terminal voltage and number of ampere-turns, one diameter or area of cross-section, which exactly satisfies the conditions of the problem ; it is practically independent of the length. When designing shunt coils, therefore, the diameter is first determined, and then the space required for the winding is considered, which upon being filled gives the required magnetizing force.

As we have already stated, the resistance of conductors increases with increase of temperature. According to the results of the experiments of Dr. Matthiessen, the resistance of metals increases with the temperature according to the empirical formula :—

$$R_t = R_0(1 + at \pm \beta t^2),$$

in which R_t = resistance at a temperature of $t°$ C.
R_0 = ,, ,, 0° C.

a and β are numerical constants depending on the metal forming the conductor.

For all but the most accurate calculations the approximation $R_t = R_0(1 + a t)$ is sufficient. For copper $a = 0{\cdot}00387$ and $\beta =$

0·00000 597. In other words, the resistance of copper increases with increase of temperature at the rate of 0·387 per cent. for each degree centigrade increase. If the Fahrenheit scale be used, the value of $a = 0{\cdot}00216$—i.e. the increase in resistance is 0·216 per cent. for each degree Fahrenheit increase of temperature. These numbers, 0·00387 and 0·00216, are known as the temperature coefficient, which may be defined as that constant by which the resistance of a wire at $t°$ must be multiplied to get the increase on the resistance at $(t+1)°$.

We are now in a position to determine the value of k, used in the formulae on page 239. If the normal working temperature of a field magnet coil is $t°$ above 15° C., then

$$k = (1 + at) = (1 + {\cdot}00387\,t)$$

and we have for formulae (page 239) for a temperature of $(15+t)°$ C.:

$$A = \frac{({\cdot}6756 \times 10^{-6})(1 + {\cdot}00387\,t)\,l_m CS}{E_m}$$

and
$$\delta = {\cdot}0009275\sqrt{\frac{(1 + {\cdot}00387\,t)\,l_m CS}{E_m}}$$

From the equation $CL = L \times \frac{E_m}{R_m}$ we may deduce several important relationships by introducing the resistivity (specific resistance per centimetre cube) at the normal working temperature. Let this resistivity be σ_1, and since

$$R_m = \frac{\sigma_1 L}{A}$$

we get, by substitution,

$$CL = L \times \frac{E_m}{\frac{\sigma_1 L}{A}} = \frac{AE_m}{\sigma_1},$$

from which we get

$$\frac{A}{C} = \frac{L\sigma_1}{E_m}$$

$$= \text{cross-section of wire per unit current,}$$

and
$$\frac{C}{A} = \frac{E_m}{L\sigma_1} = \text{current-density.}$$

At this point we may refer to the American system of using the *circular-mil* as the unit of cross-sectional area, which tends both to convenience and simplicity by obviating the use of $\frac{\pi}{4}$ as a factor in the connexion between the area and diameter of a circular wire. Thus a wire one mil ($\frac{1}{1000}$th of an inch) in diameter has an area of

one circular-mil, and the area of any circular wire in circular-mils is found by simply squaring the diameter (in mils). In connexion with this system of units, it is usual to take the resistance of one mil-foot of the wire as the resistivity of the wire. At a temperature of 60° C. the resistance of one mil-foot of copper is approximately 12 ohms.

In this system of units, the current-density is

$$\frac{C}{d^2} = \frac{1}{12} \times \frac{E_m}{L}.$$

Therefore, with a given voltage, E_m, at the terminals of the field magnets, the current-density is inversely proportional to the length of wire used in the coils. Let C_d denote current-density, i. e. amperes per circular-mil, and we have

$$L = \frac{1}{12} \times \frac{E_m}{C_d}$$

which indicates that by limiting the value of the current-density we may limit the length of wire used. Thus, with a minimum current-density of 0·0006 ampere per circular-mil (say, 850 amperes per square inch), we have

$$L = \frac{1}{12} \times \frac{E_m}{\cdot 0006} = 138\,E_m.$$

And with a maximum current-density of 0·003 ampere per circular-mil (say, 4250 amperes per square inch), we have

$$L = \frac{1}{12} \times \frac{E_m}{\cdot 003} = 27{\cdot}8\,E_m.$$

That is, the length of the wire varies from 27·8 to 138 feet per volt, or 27·8 to 138 times the terminal voltage of the field magnet coils. Again, we have

$$d^2 = \frac{12\,LC}{E_m}$$

but

$$L = S\,l_m$$

$$\therefore \quad d^2 = \frac{12\,CS\,l_m}{E_m}$$

and

$$\left.\begin{aligned} d\,(\text{mils}) &= \sqrt{\frac{12\,CS\,l_m}{E_m}} \\ &= 2\sqrt{3}\,\sqrt{\text{ampere-feet per volt}} \end{aligned}\right\}$$

Dividing this result by 1000, we, of course, obtain the diameter in inches.

§ 53. **The Weight of Wire forming Field Magnet Coils.** The method of procedure which must be followed when it is required to find the weight of the magnet winding depends largely upon the factors which are given or known. Thus, if we know the gauge and the total length of wire used, we may use either of the following approximate formulae:

$$W = 3{\cdot}02\, d^2\, L \text{ pounds}$$

where d is in inches and L in feet;

or
$$W = \frac{d^2}{62{\cdot}57} \text{ pounds per mile}$$

where d is the diameter in mils.

Also pounds per yard = area in square inches × 11·5625. Or the weight may be taken from the table given on p. 229, which gives the weight for 1000 yards of the different sizes.

§ 54. **Loss of Power in Field Magnet Winding.** Very often the winding has to be such that with a definite magnetizing power there is a limit fixed as to the number of watts which may be wasted as heat. Thus, if the heat waste be P_m watts, the terminal voltage E_m, and if the magnetizing current be C amperes, we have by Ohm's Law:

$$E_m = C \times R_m = \frac{P_m}{E_m} \times R_m$$

a relationship which takes into account the normal working temperature.

But
$$\frac{C}{d^2} = \frac{1}{12} \times \frac{E_m}{L}$$

and
$$d^2 = \frac{12\,LC}{E_m}$$

$$= \frac{12\,CS\,l_m}{E_m} \qquad \because\ L = S\,l_m$$

$$= 12\,\frac{CS\,l_m}{\dfrac{P_m\,R_m}{E_m}} = \frac{12\,CS\,l_m}{P_m} \times \frac{E_m}{R_m}$$

$$= \frac{12\,C^2 S\,l_m}{P_m}$$

and from the above we get as the weight of L feet of copper d mils in diameter:

$$W = \frac{d^2}{62{\cdot}57} \times \frac{L}{5280} \text{ pounds,}$$

and substituting the above value of d^2 we obtain :

$$W = \frac{12 \times C^2 S\, l_m L}{62{\cdot}57 \times 5280 \times P_m}$$

$$= \frac{(CS \times l_m)^2}{27530{\cdot}8 \times P_m} = \frac{36{\cdot}32}{1000000} \times \frac{(CS \times l_m)^2}{P_m}$$

$$= 36{\cdot}32 \times \frac{\left(\frac{CS \times l_m}{1000}\right)^2}{P_m} \text{ pounds.}$$

The following notes bearing on this part of the subject, taken from Thompson's 'Dynamo-Machinery,' will also be found useful :

$$CS = k_1 L \sqrt{h}$$

where h = depth of winding in inches.

L = length of wire in inches.

k_1 = a coefficient which depends on the gauge of wire and thickness of insulation.

Also $$W = k_2 \frac{p}{L} \sqrt{\frac{CS}{1000}} \text{ pounds}$$

where k_2 = a second coefficient varying with the gauge of wire.

p = perimeter of the coil in inches.

The above formulae are applicable to cases where a temperature limit is imposed, $2\frac{1}{2}$ square inches being allowed per watt. If no such limit is assumed, it is more convenient to replace them by the following formulae :

$$CS = k_3 \sqrt{WLh \div p}$$

and $$W = k_4 \times \frac{p^2}{L} \times \left(\frac{CS}{1000}\right)^2 \text{ pounds.}$$

The four numerical coefficients, k_1, k_2, k_3, k_4, have the following values :

S.W.G.	Diam. of bare wire in mils.	k_1	k_2	k_3	k_4
19	40	522	0·495	820	0·195
$10\frac{1}{2}$	120	542	0·520	850	0·205
$5\frac{1}{2}$	200	570	0·615	900	0·246

§ 55. The Permissible Heating of Magnet Coils and Surface of Emission. As is well known, the passage of electricity through a wire is accompanied by the generation of heat, with the result that a general rise of temperature in the wire is produced, which goes on until the rate of dissipation of heat by radiation is equal to the rate of generation. This is an example of Joule's effect, and, according

to Joule's law, the amount of heat developed per second in the field magnet coils of a dynamo, or, in other words, the energy wasted in the magnet coils, is given by the product of the square of the magnetizing current into the resistance of the magnet coils, and it is clear that the maximum rise of temperature at the outside of the magnetic coils above that of the surrounding air is a function of the ratio existing between the rate of generation of heat in the coils to the cooling or exposed surface of the coils. By the cooling surface we mean that portion of the superficial area of the coils which is in direct contact with the surrounding air, and is, in fact, radiating surface.

It may be proved by experiment that an increase of the temperature in the coils causes an increased radiation of heat per degree rise in temperature, but the ratio of increase diminishes as the temperature increases, and an increase of the amount of heat generated in the coils increases their temperature, but less than proportionately.

It is, therefore, necessary in practice to fix the permissible heating of magnet coils and surface of emission, so as to prevent overheating of the coils by providing a cooling or radiating surface proportional to the heat developed in the coils, i. e. to the energy wasted in the coils as heat. We must now consider the assigned temperature limit, which experience has shown to be compatible with efficiency and safety, and it will be convenient to term *the energy dissipated per unit of radiating surface in watts the Specific Energy Loss.* Under ordinary conditions the temperature increase per unit of specific energy loss is of the nature of a constant, which we shall denote by θ_m, so that the actual increase of the temperature of any magnet coil is given by the formula:

$$\theta = \theta_m \times \frac{P_m}{A_m}$$

where θ = rise of temperature of the coils above the atmospheric temperature.

P_m = energy in watts wasted in the coils.

A_m = cooling or radiating surface.

$\therefore \frac{P_m}{A_m}$ = specific energy loss.

Now, Esson finds that in practice the emission of heat from field magnet coils is at the rate of $\frac{1}{355}$ watt per square centimetre per degree Centigrade; and we have therefore

$\theta_m = 355$ for the Centigrade scale and cooling area in square centimetres.

$= 55$ for the Centigrade scale and cooling area in square inches.

$= 99$ for the Fahrenheit scale and cooling area in square inches.

If we take the rise of temperature of magnet coils to vary in practice between the limits of 10° C. and 50° C., we find, using Esson's value for θ_m, that

$$\text{for a rise of } 10^\circ\,\text{C.}, \frac{P_m}{A_m} = \frac{10}{55} = \frac{2}{11} \text{ watt per sq. in.}$$

$$\text{and for a rise of } 50^\circ\,\text{C.}, \frac{P_m}{A_m} = \frac{50}{55} = \frac{10}{11} \text{ watt per sq in.,}$$

giving as a mean value $\frac{6}{11}$ watt per square inch, for, say, a temperature rise of 30° C. In other words, the radiating surface for the temperature rises mentioned varies from 5·5 to 1·1 square inches per watt (35 to 7 square centimetres per watt), giving as a mean value 3·3 square inches per watt. This means, that with a certain coil space permitting only a certain amount of surface, the winding must be so chosen that it dissipates energy at the rate of one watt for each 3·3 square inches of radiating surface, so that the temperature shall not rise above 30° C.

It may be mentioned here that Kapp allows 2·5 square inches (16·2 square centimetres) per watt dissipated in the field magnet coils, whilst in a Brush arc dynamo two square inches of radiating surface were allowed for each watt lost. Dr. S. P. Thompson gives the following formula to find the maximum permissible current, if the rise of temperature t is prescribed as a limit:

$$\text{Maximum permissible current} = \sqrt{\frac{t^\circ\text{C.} \times \text{sq. cm.}}{355 \times \text{resistance (hot)}}},$$

or

$$\text{Maximum permissible current} = \sqrt{\frac{t^\circ\,\text{F.} \times \text{sq. inches}}{99 \times \text{resistance (hot)}}}.$$

'If we assume that a safe limit of temperature is 90° F. (or 50° C.) higher than the surrounding air, then the largest current which may be used with a given electromagnet is expressed by the formula:

$$\text{Highest permissible amperage} = 0{\cdot}95\sqrt{\frac{\text{sq. inches}}{\text{resistance (ohms)}}}.$$

Similarly, for *shunt coils* we have:

$$\text{Highest permissible voltage} = 0{\cdot}95\sqrt{(\text{sq. inches}) \times (\text{resistance})}.\text{'}$$

In an ordinary bi-polar dynamo (see Hawkins and Wallis's *The Dynamo*) of about 20 kilowatts output, with field coils $1\frac{1}{2}$ inches

deep, and the maximum temperature attained by the outside of the coils about 110° F. (surrounding air at 60° F.), the resistance of the coils at the higher temperature (110° F.) is to that of the lower temperature as 115 : 100.

For a coil of given volume the energy wasted is the same for the same magnetizing power, irrespective of the gauge of the wire.

If H = amount of heat generated per second,

$$H = C^2 R_m,$$

and the excitation = CS ampere-turns.

Now, if the gauge be changed so that for the same volume we have the same magnetizing power, or excitation, given by S_1 turns, and C_1 amperes, then

$$CS = C_1 S_1$$

and
$$C_1 = \frac{S}{S_1} \times C$$

but
$$\frac{R^1_m}{R_m} = \frac{S_1^2}{S^2} \text{ (p. 233)}$$

or
$$R^1_m = \frac{S_1^2}{S^2} \times R_m$$

and the heat now generated is

$$\begin{aligned} H_1 &= C_1^2 R^1_m \\ &= \left(\frac{S}{S_1} \times C\right)^2 \times \left(\frac{S_1^2}{S^2} \times R_m\right) \\ &= C^2 R_m \text{ as before.} \end{aligned}$$

Worked Example. It is required to determine the size, weight, and resistance of the copper wire forming the field magnet coil of a $2\frac{1}{2}$ kilowatt series-wound dynamo (of the single magnet type) giving 10 amperes at 250 volts. We may assume that 5200 ampere-turns are required for the excitation, and that the core is 4 inches by $2\frac{1}{4}$ inches, and also 8 inches long. The depth of the winding h is 1·3 inches.

This is given as an example of the case where for a definite number of ampere-turns of magnetizing force, which are given, the magnetizing current is fixed, i. e. is 10 amperes. The size of the wire is obviously, to some extent, fixed by the current which has to traverse it, and by the permissible heating. We shall assume that the temperature increase of the coils has not to exceed 30° C. at normal load.

Dividing the number of ampere-turns by the magnetizing current gives the number of turns of wire forming the coil, thus

$$\begin{aligned} S_{se} &= \frac{C_m S_{se}}{C_m} = \frac{5200}{10} \\ &= 520 \text{ turns.} \end{aligned}$$

Since the cross-sectional area of the core is known, we may determine the

total length of wire used from the length of a mean turn, which is found as follows.

The length of a mean turn is

$$l_m = 2 \times [(4+1\cdot3)+(2\cdot25+1\cdot3)]$$
$$= 2 \times 8\cdot85 = 17\cdot7 \text{ inches,}$$

and the total length of wire used is

$$L = \frac{17\cdot7 \times 520}{12} = 767 \text{ feet.}$$

Since the permissible heating is fixed for an increase of 30° C., we have next to determine the radiating surface of the coil; let A_m = radiating surface, then

$$A_m = 8 \times [2 \times (4+2 \times 1\cdot3)+2 \times (2\cdot25+2 \times 1\cdot3)]$$
$$= 183 \text{ square inches.}$$

If we allow 3·3 square inches of radiating surface for the dissipation of energy at the rate of one watt, we have

$$C^2_m R_m = \frac{183}{3\cdot3},$$

and

$$R_m = \frac{183}{100 \times 3\cdot3} = 0\cdot554 \text{ ohm.}$$

This, of course, is the resistance of the coil at a temperature of 30° above that of the atmosphere, which we will suppose to be at 15° C. Now, since the resistance of copper increases at the rate of 0·387 per cent. for each degree Centigrade increase of temperature, the normal resistance of the winding at 15° C. is

$$R = R_m \times \frac{1}{(1+\cdot00387 \times 30^\circ)}$$
$$= \frac{\cdot554}{1\cdot1161} = 0\cdot496 \text{ ohm.}$$

If we now divide the total length of wire forming the coil by the value of the resistance, we obtain the *specific length, as regards resistance*, or *feet per ohm*, of the wire to be used, or

$$\text{specific length} = \frac{767}{0\cdot496} = 1546,$$

and from the wire table, given on p. 238, we find the nearest gauge is No. 10 S.W.G. The diameter of this wire, bare, is 0·128 inch, and if we allow 12 mils for insulation, we have 0·14 inch as the diameter of the insulated wire. The number of layers will, therefore, be

$$n = \frac{1\cdot3}{0\cdot14} = 9 \text{ layers,}$$

and the number of turns per layer is

$$t = \frac{8}{0\cdot14} = 57 \text{ turns.}$$

$$\therefore \quad \text{total number of turns} = S = 57 \times 9$$
$$= 513 \text{ turns.}$$

To determine the weight of wire we may use the formula

$$W \text{ (pounds)} = 3\cdot02\, d^2 \text{ (inches)} \times L \text{ (feet),}$$

since both d and L are known.

$$\therefore \quad W = 3\cdot02 \times (0\cdot128)^2 \times 767$$
$$= 38 \text{ pounds.}$$

EXERCISES VI B.

(1) An iron ring of square section (4 cm. side) and 10 cms. mean radius has a gap of 3 mm. width cut across it. It has to be wound once over with copper wire so that when connected to 6-volt mains the field in the gap is 10000 C.G.S. lines. Determine the particulars of the winding if the permeability of the iron is 2500, and the resistance per inch cube of the copper is 0·66 microhm.

(2) Calculate the size, resistance, and weight of copper wire such that, if wound on a cylindrical iron core 6 inches long, $3\frac{1}{2}$ inches in diameter, and having a P.D. of 50 volts between its terminals, 4440 ampere-turns will be produced. The diameter of the completed coil is to be $7\frac{3}{4}$ inches. A cubic foot of copper weighs 550 pounds. (C. and G.)

(3) Calculate the size, resistance, and weight of copper wire such that, if wound on a magnet core 7 inches by $3\frac{1}{2}$ inches, and having a potential difference of 25 volts maintained between the terminals, 5000 ampere-turns will be produced. Length inside former is 8 inches. (C. and G.)

(4) Determine the diameter of a copper wire required to give 3000 ampere-turns for a field magnet coil when the terminal P.D. is 50 volts, if $l_m = 3$ feet, neglecting increase of temperature.

(5) Determine the diameter of the wire for the last example if the working temperature is 15° C. above the surrounding temperature.

(6) A field magnet coil requires 4000 ampere-turns if the internal diameter is 12 inches and the external diameter 15 inches; determine the size of the wire if the terminal P.D. is (1) 50 volts, and (2) 100 volts. Also determine (3) the diameter of the wire if the increase in temperature is 25° C., when the P.D. = 100 volts; (4) the weight of the wire in (3), if the watts wasted as heat are 200.

(7) When fully excited, the values of the induction B, in the various parts of the magnetic circuit of a dynamo, and the mean lengths of path l of the magnetic flux, are as given below:—

Magnet cores, high permeability cast steel (two in a circuit)	$l = 30$ cms.	: B = 13000
Magnet yokes, high permeability cast steel	$l = 40$,,	B = 10000
Air-gap (flux crosses twice in a circuit) .	$l = 7$ mm.	: B = 7000
Armature Teeth (flux passes through two groups each)	$l = 25$,,	B = 20000
Armature core, charcoal iron sheets . .	$l = 30$ cms.	: B = 10000

The mean length of a turn of shunt winding is 80 cms., and the E.M.F. measured at the ends of the coils on a pair of limbs is 100

volts. What would be the size of wire required for a shunt winding, neglecting any allowances for armature reaction? (C. and G.)

(8) Two bobbins of an electromagnet are wound ·026 with wire which has a resistance of 0·026 ohm per metre. The coil on each bobbin is 10 cms. long, and is 5 cms. mean diameter. If there are 5 layers of 100 turns on each limb and a current is sent through them connected in series, determine the magnetizing force if a pressure of 5 volts is maintained at the terminals of the electromagnet.

(9) The limb of a dynamo magnet is of square section, each side measuring 8 inches; the length of the portion occupied by shunt winding is 7 inches; what size and quantity of wire will be required to form a shunt coil which will give 12000 ampere-turns when a pressure of 100 volts is maintained between its two ends, and allowing a cooling surface of 2 square inches per watt wasted? The cooling surface may be taken as the area of the external surface of the coil, plus that of the two ends. The resistance of a cubic inch of copper is approximately 0·66 microhm at 60° F., and the coefficient for increase of resistance with rise of temperature 0·21 per cent. per degree F. The working temperature of the coil is to be taken at 110° F. (C. and G.)

§ 56. Attractive Force of Magnets. It will be obvious that there will be considerable attractive force exerted between the armature core and the polar surfaces of the field magnets, both in dynamos and motors, when working. This attractive force measures the tendency to bring the armature core and pole-pieces into contact, and also the pull required to separate two magnets when in contact. Consequently the consideration of tractive force in the design of electromagnets and dynamos is an important problem.

It may be proved that the attractive force exerted between each square centimetre of a pair of separated magnetized surfaces placed parallel to one another is proportional to the square of the number of lines that pass from one surface to the other; or, in other words, to the square of the induction, B_g, in the air-gap between the surfaces. Maxwell has shown, mathematically, and Bosanquet has experimentally proved, that, for a field in which the magnetic flux is uniformly distributed over an area of A sq. cm., that the total tractive force or pull between two parallel surfaces is

$$P = \frac{A B_g^2}{8\pi} \text{ dynes.}$$

This value for the pull is strictly applicable only when the distance between the surfaces is small compared with the area of the surfaces,

or when one surface extends considerably over the other. To give the tractive force in pounds, we proceed as follows :—

$$P = \frac{A \times B_g^2}{8\pi} \text{ dynes} = \frac{A B_g^2}{8\pi \times 981 \times 453 \cdot 6} \text{ pounds}$$

$$= \frac{A B_g^2}{11183000} \text{ pounds.}$$

Worked Examples. (1) Two bars, each 5 square centimetres in cross-section, have their faced ends touching within a solenoid excited so as to produce a magnetic induction of 15000 lines per square centimetre through the joint. Give, in pounds, the force requisite to pull them apart. (C. and G., 94.)

$$P = \frac{A B_g^2}{11183000} \text{ pounds}$$

$$= \frac{5 \times 15000 \times 15000}{11183000} \text{ pounds}$$

$$= 100 \cdot 6 \text{ pounds.}$$

(2) A horse-shoe magnet provided with a keeper is required to carry 389 kilogrammes. Determine the area in square inches if the induction is given as 20000.

Since the weight of one gramme is equivalent to 981 dynes,

$$389 \text{ kilogrammes} = 981 \times 1000 \times 389 \text{ dynes,}$$

and

$$981 \times 1000 \times 389 = \frac{A B_g^2}{8\pi} \times 2$$

$$= \frac{A \times 20000 \times 20000}{8 \times 3 \cdot 1416} \times 2,$$

where A is in square centimetres. The multiplier, 2, is introduced because two poles are employed.

$$\therefore \quad A = \frac{981 \times 1000 \times 389 \times 8 \times 3 \cdot 1416}{2 \times 4 \times 10^8} \text{ sq. cm.}$$

$$= 12 \cdot 26 \div 6 \cdot 45 \text{ sq. inches}$$

$$= 1 \cdot 9 \text{ sq. inches.}$$

(3) Determine the number of ampere-turns required to excite an electromagnet, given that the area of cross-section of the wrought-iron limbs and armature is 12·7 sq. cm., $l_m = 30$ cm., $l_a = 12 \cdot 7$ cm., and $l_g = 1 \cdot 27$ cm. The electromagnet is required to exert a pull of 60 kilogrammes upon its armature, and the coefficient of leakage may be assumed to be 1·5.

(a) *Determination of* B_g.

From the data given

$$2 \times \frac{B_g^2 \times A}{8\pi} = 60 \times 1000 \times 981,$$

$$\therefore \quad B_g^2 = \frac{60000 \times 981 \times 8 \times 3 \cdot 1416}{2 \times 12 \cdot 7}$$

and

$$B_g = 7632.$$

(b) *Determination of ampere-turns.*

This value of B_g is the induction in the armature, and from the magnetization curve or table for wrought iron, given on page 208, μ_1 may be taken as 2500. And since the coefficient of leakage is 1·5, the induction for the

magnet limbs $= 1{\cdot}5 \times 7632 = 11448,$

for which $\mu_2 = 1550;$

and since
$$\mathrm{N} = \mathrm{B} \times \mathrm{A} = \frac{1{\cdot}257 \times \mathrm{CS}}{\dfrac{l_m}{\mathrm{A_m}\,\mu_2} + \dfrac{2l_g}{\mathrm{A}} + \dfrac{l_a}{\mathrm{A_a}\,\mu_1}},$$

$$\therefore \quad 7632 \times 12{\cdot}7 = \frac{1{\cdot}257 \times \mathrm{CS}}{\dfrac{30}{12{\cdot}7 \times 1550} + \dfrac{2 \times 1{\cdot}27}{12{\cdot}7} + \dfrac{12{\cdot}7}{12{\cdot}7 \times 2500}},$$

$$\therefore \quad \mathrm{CS} = \frac{7632 \times 12{\cdot}7}{1{\cdot}257}\left(\frac{30}{12{\cdot}7 \times 1550} + \frac{2{\cdot}54}{12{\cdot}7} + \frac{12{\cdot}7}{12{\cdot}7 \times 2500}\right)$$

$$= \frac{7632}{1{\cdot}257}\left(\frac{3}{155} + 2{\cdot}54 + \frac{12{\cdot}7}{2500}\right)$$

$$= \frac{7632}{1{\cdot}257} \times 2{\cdot}564$$

$= 15567$ ampere-turns.

EXERCISES VIc.

(1) In the equation $\mathrm{P} = \dfrac{\mathrm{A}\mathrm{B}_g^{\,2}}{\mathrm{K}}$ prove that $\mathrm{K} = 72134000$, when P is given in pounds, A in sq. inches, and B_g in C.G.S. lines per sq. inch.

(2) A horse-shoe electromagnet is made of round annealed wrought iron, 0·5 inch in diameter. When excited by means of 392·5 ampere-turns, the value of μ may be taken as 907. Determine the load which the two poles will carry if the complete length of the magnet and keeper is 20 inches.

(3) A horse-shoe electromagnet with a core and keeper forged from 1-inch square iron is excited by 300 ampere-turns. Let the joints between the pole faces and keeper be scraped so as to make a perfect fit, and assume that the permeability of the metal at the junction produced by 300 ampere-turns is 1500. Length of magnetic circuit 16 inches. Find the total flux through core and keeper, and the force required to tear the keeper off. (C. and G., 93.)

(4) A horse-shoe magnet has a cross-sectional area of 15 sq. inches and a length of 20 inches; the armature is 14 inches long. Determine the load which the electromagnet will carry, if made of wrought iron, and the armature has the same sectional area as the limbs, when the flux density is 16000.

(5) Professor S. P. Thompson, D.Sc., fixes the economical limit of 150 pounds per square inch for tractive purposes. Determine the induction per sq. cm. and also per sq. inch for this value.

CHAPTER VII

THE GENERATION OF ELECTRICAL PRESSURES

Section I. The Principles of Electromagnetic Induction.

§ **57. Faraday's Discovery.** As is well known, the transformation of mechanical energy into electrical energy is, of all the known methods of generating electricity, the one upon which we depend for the production of currents of electricity commercially. Furthermore, the term *electric generator* is now reserved to designate *a machine for converting mechanical energy into electrical energy by virtue of continuous relative motion taking place between electrical conductors and a magnetic field or fields, whenever such motion causes a change to occur in the amount of the magnetic flux linked with the conductors.* All generators possess two essential parts: (1) a system of electrical conductors forming the armature, and (2) a system of magnets designed to produce a strong magnetic field in which the conductors forming the armature are disposed.

The two parts are so arranged that when relative motion takes place between them the phenomenon of electromagnetic induction is developed, the principles of which were discovered by Faraday in 1831. Whenever the phenomenon of induction is realized by such an arrangement the conductors are said to be the seat of an electromotive force (E.M.F.).

The principles of electromagnetic induction as announced by Faraday may be stated in general terms as follows:

An E.M.F. is induced in a conductor whenever relative motion takes place between the conductor and a magnetic field or fields, so that the number of lines of magnetic force looped or linked with the conductor vary. This phenomenon was termed *electromagnetic induction* by Faraday, who appears to have grasped the principle of reversibility, and to have clearly foreseen that many of the effects of which electricity is the cause may in their turn be utilized to produce currents of electricity. With wonderful foresight, simple deduction, and reasoning, he concluded correctly that inasmuch as the passage of electricity through conductors radiates electromagnetic wave systems and sets up magnetic fields in the space surrounding conductors carrying a current, the reverse process should set up an

electromotive-force—that is, the setting up of a magnetic field about a conductor should give rise to an induced E.M.F. in that conductor. Not only were his conclusions exact, but his experiments conclusively proved that induced currents result in such cases when the conductor formed part of a closed circuit.

There are many ways of producing these interactions between magnetic fields and conductors which result in the production of induced E.M.F.'s; but it will be more interesting if we refer to Faraday's experiments on electromagnetic induction as an introduction to the methods of producing induced currents by dynamo-machinery. If, for instance, a hollow helix or coil of many turns of insulated wire (Fig. 47) be connected with a delicate galvanometer, so as to form a closed circuit, it will be found that the galvanometer indicates a transient current when a strong bar magnet is quickly inserted into the coil. The insertion of the magnet within the coil introduces a magnetic field into the space surrounding the coil, and the number of lines of force embraced by the coil is increased, and the space is put into that peculiar state of strain which constitutes a magnetic field, with the result that another state of stress is produced in the wire which is termed electromotive force; it is an induced electrical pressure which causes a current to traverse it so long as the change is taking place. But when the magnet and coil are stationary relatively to one another, no E.M.F. is induced. Upon withdrawing the magnet (i. e. diminishing the flux enclosed) another momentary E.M.F. and current, opposite in direction (as indicated by the galvanometer) to the former are induced. If the magnet be fixed in position, and the coil is moved relatively to the magnet, exactly similar results are obtained; or both the magnet and the coil may simultaneously take part in the movements, and it is found that in all cases the more rapid the motion the greater will be the E.M.F. and current induced. The energy corresponding to the passage of these induced currents is due to the mechanical energy spent in producing the motion, and the phenomenon is an example of the conversion of mechanical energy into electrical energy. Increasing the number of

FIG. 47.

lines of force linked with the conductor gives rise to an E.M.F. in one direction, whilst decreasing the number of lines of force linked with the conductor sets up an E.M.F. in an opposite direction.

Since a conductor traversed by a current is surrounded by a magnetic field, and a coil carrying a current is equivalent to a magnet, it follows, as Faraday's experiments proved, that currents may be induced by neighbouring currents. If the magnet in the above experiment be replaced by a coil traversed by a current, and relative motion takes place between the two coils, that is, when one coil approaches to or recedes from the other, or, if the two coils be fixed permanently, and a current be started, stopped, or varied in magnitude in one of them, an E.M.F. and current is induced in the neighbouring coil.

The determination of the direction of the electromotive-force induced in a conductor moving across a magnetic field is a simple matter when we remember that work is done whilst the change in the flux embraced by a coil is taking place, for this implies that the motion of a conductor across a magnetic field is resisted by some force acting in the opposite direction to that of the applied force doing work. In other words, the direction of the induced current in the closed coil is such as to produce motion opposite to that possessed by the coil, in virtue of the electrodynamic forces set up (see p. 288) as the result of the interaction between the current induced and the magnetic flux. This follows, as a matter of course, from the Principle of the Conservation of Energy, and when the motion of the coil is such as to produce a diminution of the number of lines of force enclosed, the direction of the induced current is such that the lines of force set up by the induced current tend to increase the number of the lines of force enclosed by the coil, and thus oppose the change in the number of the lines of force taking place. Similarly, if the motion produces an increase in the flux enclosed the direction of the induced current is such that the lines of force set up are opposite in direction to those of the field, and tend to diminish the number of the lines of force enclosed. Induced currents thus resist (1) the motion, and (2) the change in the magnetic flux. This principle was first announced by Lenz, and the following simple statement is known as Lenz's Law: 'The direction of an induced current is in all cases such that by its electromagnetic action it tends to oppose the motion which produces it.'

Both Lord Kelvin and Helmholtz have deduced a general law of induction from the Principle of the Conservation of Energy, which may be stated as follows: 'A circuit traversed by a variable magnetic flux is the seat of an electromotive force of induction which is in

every case equal and of contrary sign to the rate of variation of the flux with respect to the time.'

Since it is a matter of much importance in practice to be able to determine the direction of the E.M.F. induced in dynamo machinery, Prof. Fleming has devised and introduced the well-known and useful hand rule, which is as follows: Hold the forefinger, the middle finger, and the thumb of the *right hand* in such a way that they are as nearly as possible at right angles to each other; then if the forefinger point in, or represent, the direction of the magnetic flux (Fore and Flux), and the thumb the direction of the motion of the conductor (thuMb and Motion), the direction of the middle finger will represent the direction of the induced current (mIddle and Induced) as shown in Fig. 48.

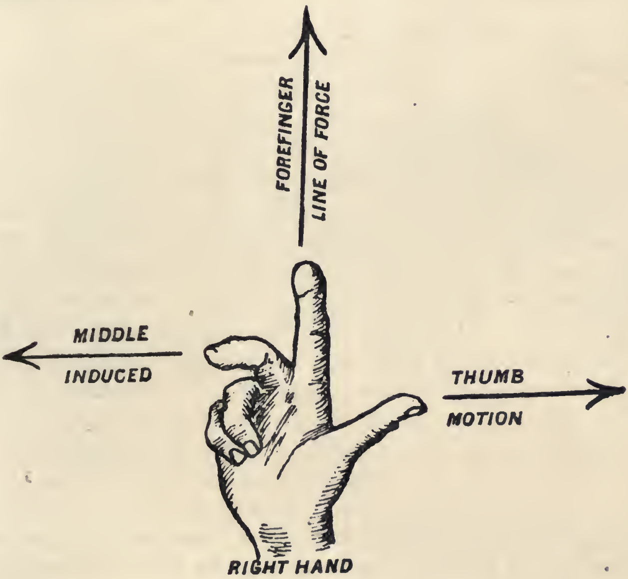


FIG. 48.

Collecting together the principles explained above, we have the following simple statements as the fundamental laws of electromagnetic induction:—(1) A *decrease* in the number of the lines of force linked with a closed circuit induces a current in the conductor in the *positive* direction (i. e. a *direct* current producing lines of force coinciding in direction with the field), and an *increase* in the number of lines looped with a closed circuit induces a current in the conductor in the *negative* direction (i. e. produces an *inverse* current). (2) The

induced E.M.F. acting in a closed circuit is equal to *the time rate of change in the number of linkages.*

By the term linkages is meant the number of lines of force linked or looped with the circuit: thus, if the total number of lines of force enclosed by a single turn of wire be N, then N = the number of linkages; whilst if the circuit enclosing a flux of N lines consist of n turns, then n N = the number of linkages. It is obvious, therefore, that a circuit traversed by a variable magnetic flux is the seat of an electromotive force of induction, which is in every case equal and of contrary sign to the rate of variation of the flux with respect to the time, and the direction of the E.M.F. is readily given by Lenz's Law, which is as follows: The direction of an induced current is in all cases such that by its electromagnetic action it tends to oppose the motion which produces it.

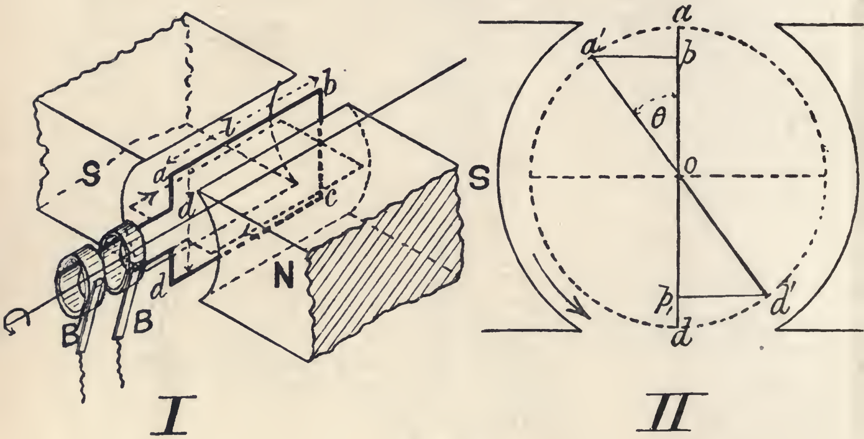


FIG. 49.

To understand how alternating E.M.F.'s are produced in practice, it will be well to consider the working of an ideal alternator, which we may consider to consist of a single rectangular coil of wire, capable of being rotated about a longitudinal axis in a uniform magnetic field formed by the polepieces N and S, as shown in Fig. 49.

As shown in the figure, the axis is perpendicular to the direction of the lines of force, and the magnetic flux linked with coil is made to vary from instant to instant by rotating the coil at a uniform speed. Let H denote the intensity of the uniform magnetic field, l the length of the coil in centimetres, and d the width. Then if

$$d \, . \, l = \text{S square centimetres,}$$

and $$\text{S H} = \text{N} = \text{the maximum number of linkages}$$

when the coil is vertical, the variation in the number of linkage per revolution of the coil is 4 N. We shall also suppose that the coil $a\,b\,c\,d$ is not closed upon itself, but that the ends are connected to two contact-rings B and B_1, upon which press metallic brushes and make connexion with an external circuit, which is perfectly simple and does not contain anything which would set up secondary effects. In the normal position as shown, the sides $a\,d$ and $b\,c$ are vertical, and the coil encloses N, the maximum number of lines of force. When the coil has rotated through 90°, the sides $a\,d$ and $b\,c$ are horizontal, and the plane of the coil is parallel to the lines of force. In this position the coil encloses the minimum number of lines of force, which is zero; obviously the rotation through a right angle is equivalent to emptying the coil of lines, to do which each of the sides $a\,b$ and $c\,d$ must cut $\frac{1}{2}$ N lines of force. During this rotation the sides $a\,b$ and $c\,d$ are the seats of E.M.F.'s, the directions of which are indicated by the arrow-heads. By considering the rotation through the next quadrant, we see that the coil takes up a vertical position, and N lines are again enclosed; in other words, the movement through the second quadrant is equivalent to filling the coil with lines, and since the magnetic flux enclosed is varied, the sides $a\,b$ and $c\,d$ are again the seats of E.M.F.'s, the directions of which are the same as in the first quadrant, as may be shown by applying Lenz's Law. Since the lines of force enter the coil from the opposite side during the rotation through the second quadrant, the change in the flux enclosed is $-$N. On continuing the rotation through the next two quadrants, the coil is again emptied and filled as before, but in a negative sense, and the direction of the resulting E.M.F. is reversed. Thus, in each complete revolution of the coil, the induced E.M.F. is reversed twice in direction, and the contact-ring B supplies current *to* the external circuit during the first half-revolution, and B_1 does the same during the second half-revolution. In other words, this simple arrangement generates periodic electro-motive-forces, which alternate in direction twice each revolution, and the contact-rings B and B_1 are alternately positive and negative, with the result that an alternating current is supplied to the external circuit. Furthermore, it may be shown that the variation in the magnitude of the E.M.F. induced is *harmonic* or *sinusoidal*—i. e. the value of the E.M.F. is, at any instant, a *sine-function of the time.* We know, for instance, from Faraday's 'Laws of Induction' that the E.M.F. induced in the rotating coil of our ideal alternator is *proportional to the time-rate of change of the magnetic flux* threading the coil, $a\ b\ c\ d$, and a moment's consideration will suffice to prove that the following statements apply and indicate that we are dealing with harmonic functions :—

(1) The corners a and d of the rectangular coil $a\ b\ c\ d$ (Fig. 49) describe a circle with uniform velocity whilst the coil revolves in the uniform magnetic field.

(2) The points p and p_1 (the projections of a and d respectively upon $a\ o\ d$, Fig. 49, II) have a to-and-fro motion along $a\ o\ d$, whilst a and d traverse the circle, consequently p and p_1 make a complete oscillation along $a\ o\ d$ whilst a and d make one revolution, and the velocities of p and p_1 are equal, but vary from instant to instant.

(3) The length of the magnetic flux enclosed by the coil is at every instant equal to l, but the *width* of the flux enclosed varies from instant to instant from a maximum value d to a minimum value zero. Obviously the *width* at any instant is given by the length, $p\ p_1$ (Fig. 49, II), of the projection of $a'\ d'$ upon $a\ o\ d$ at the instant under consideration. Therefore the flux enclosed by the coil at any instant is proportional to the projection of $a'\ d'$ (at the instant under consideration) upon $a\ o\ d$, or, algebraically:

$$\mathsf{N}_t = \mathsf{N} \cos \theta,$$

where N = maximum flux enclosed, N_t = flux enclosed when the corner a has described the angle θ.

(4) The *rate of variation* in the flux enclosed is at any instant proportional to the rate of change in the velocities of p and p_1 along $a\ o\ d$. But the velocities of p and p_1 may readily be shown to be an example of simple harmonic motion, and at any instant to be equal to the product of the velocity of a (or d) into the sine of the angle made by $a'\ d'$ with the normal position $a\ d$; consequently the *time rate of change in the flux enclosed, or the change in the number of linkages, is proportional to the sine of an angle proportional to the time.*

(5) The *variation* in the instantaneous values of the E.M.F. induced in the coil is proportional to the *rate of variation* in the flux enclosed; it is therefore *proportional to the sine of an angle proportional to the time,* or algebraically:

$$e_t = \mathrm{E} \sin \theta,$$

where e_t = the value of the E.M.F. induced at the instant t, E = the maximum value of the E.M.F. induced, and $\theta = \omega\ t$, ω being the angular velocity of the point a.

Such is the character of the alternating E.M.F. induced in the coil, and it is obvious that an alternating E.M.F. is a harmonic quantity. We may also point out here that the same result is produced if the uniform and constant magnetic flux be replaced by an alternating magnetic flux, and the revolving coil by a fixed one, so that the coil is alternately emptied and filled as is the case with transformers. With some alternators, for instance, the magnetic poles and field

revolve, and the armature coils are stationary. If the coil of our ideal alternator consist of n turns of wire, then, obviously, the rate of change in the number of linkages will be increased n times, and, therefore, the value of the E.M.F. will be n times as large.

To produce a continuous current the two contact rings of the ideal alternator are replaced by a commutator, by means of which and collecting brushes the alternating character of the induced E.M.F. is rectified and made uni-directed, so that upon closing the external circuit a continuous current is obtained.

The laws of electromagnetic induction may be stated algebraically as follows:—

Let N_1 = the magnetic flux enclosed by the coil of the ideal alternator at the commencement of the short period of time t.

N_2 = magnetic flux enclosed at the end of the short interval of time t.

Then if E = average induced E.M.F. in absolute units:

$$E = \frac{N_1 - N_2}{t}; \text{ or, if } N_2 > N_1$$

$$E = \frac{N_2 - N_1}{t} = -\left(\frac{N_1 - N_2}{t}\right) = \text{an inverse E.M.F.}$$

These two expressions refer respectively to the two cases of a decrease and an increase in the magnetic flux enclosed by the coil. The result is more correct if we make the interval of time very small, and if δN denotes the change in the magnetic flux linked with the coil during the very small interval of time δt, then the instantaneous value of the E.M.F. induced in absolute units is:

$$e = -\frac{\delta N}{\delta t} = \text{time rate of change in the flux.}$$

The negative sign indicates that the direction of the induced E.M.F. is such that it opposes the change in the magnetic flux linked with the circuit, according to Lenz's Law.

If R is the resistance of the complete closed circuit, then the resulting current in amperes is:

$$C = -\frac{\delta N}{\delta t} \times \frac{1}{10^8} \times \frac{1}{R}$$

since there are 100000000 or 10^8 absolute units of E.M.F. in one volt.

If the rotating coil consists of S turns or spires, instead of one turn, the maximum magnetic flux linked with each turn of the coil is N, and the E.M.F. induced in each turn is added to give the total E.M.F. induced, consequently:

total E.M.F. induced in coil of s turns

$$= - \mathrm{S}.\frac{\delta \mathsf{N}}{\delta t}$$

$$= - \frac{\delta (\mathrm{S}\,\mathsf{N})}{\delta t}$$

where $\mathrm{S}\,\mathsf{N}$ = maximum number of linkages.

Section II. Classification of Armatures.

§ 58. **Armatures.** In this section only closed-coil armatures will be considered, and it is obvious that coils of insulated wire arranged to rotate in a magnetic field may be disposed in many different ways;

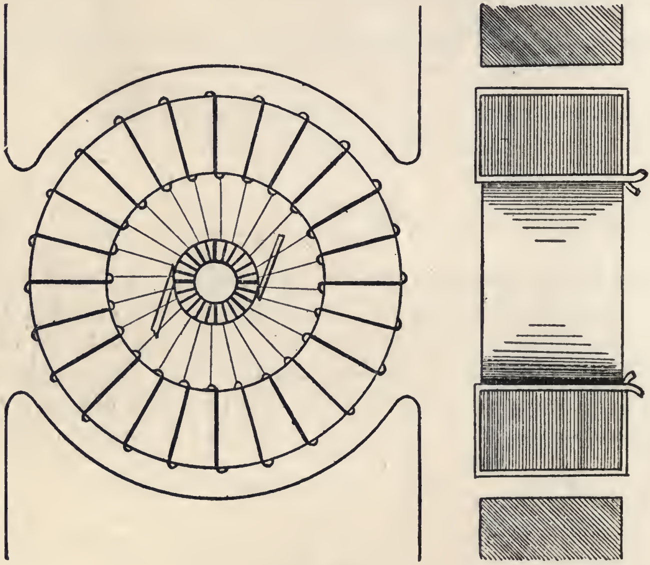


Fig. 50.

consequently there are several distinct types of armatures, and one type of armature is distinguished from the others simply by the arrangement of its conductors. Continuous current armatures may be divided broadly into three classes; i. e.

(1) *Drum Armatures*, in which the insulated conductors are wound longitudinally upon the surface of a drum or cylinder;

(2) *Ring Armatures*, in which the insulated conductors are wound spirally around a ring-shaped or hollow cylindrical core;

(3) *Disc Armatures*, in which the insulated conductors are placed

radially upon the surface of a disc-shaped frame, thus forming flat coils having their axes parallel to the shaft.

Armatures may also be distinguished as follows:

(a) *Smooth Core Armatures*, in which the conductors are exterior to the iron core;

(b) *Toothed-Core Armatures*, in which the conductors are imbedded in slots or channels, provided upon the surface of the core;

(c) *Perforated or Hole-wound Armatures*, in which the conductors are drawn through holes or ducts, extending near the surface of, but entirely within, the iron body of the armature core.

At this point it may be mentioned in passing that the iron cores of all armatures are laminated to diminish to a minimum losses by heating, Foucault's currents, and hysteresis. The thin discs of iron are thoroughly insulated from each other by varnish, asbestos and other kinds of paper, &c. The construction of armature cores is indicated in Figs. 50 and 51.

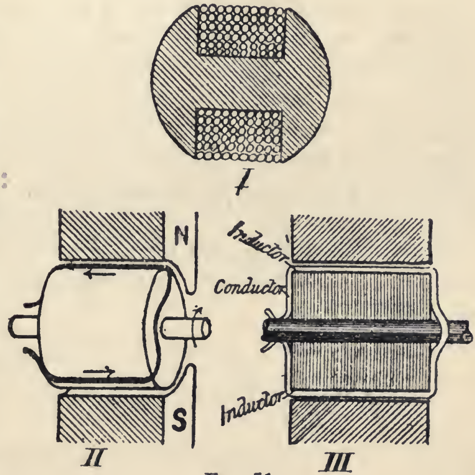


Fig. 51.

Probably the first practical armature was that known as the *shuttle* or H armature invented by Werner Siemens in 1856, and shown in Fig. 51 *I*. This type, which is now obsolete, except in magneto machines, was the forerunner of the drum type first introduced by Hefner von Alteneck in 1873. Previous to the introduction of the drum armature, however, Paccinotti in 1864 invented the Paccinotti core, in which the insulated conductors of the armature were wound in grooves which were formed by external and internal projections, so that the Paccinotti armature was an example of a toothed-core ring armature. This type of ring armature was re-invented by Gramme, who introduced it in a commercial form in 1871. Figs. 50 and 51 illustrate the ring and drum types of armatures respectively.

It is also important to notice that if we term that part of a rotating conductor which is the seat of an E.M.F. an *inductor*, then each coil of a drum armature has two inductors, whilst each coil

of a ring armature has only one inductor. The number of inductors per coil is therefore characteristic of the type of coil, and we have here an essential point of difference between drum and ring armatures. The drum-winding has a much larger proportion of *active* wire per convolution than the ring armature. There are, however, other factors to consider before one can decide the superiority, or otherwise, of one type over that of the other.

Section III. Magnitude of E.M.F. generated.

§ 59. Determination of E.M.F. set up in an Armature. For commercial purposes the electrical pressures required are large compared with the E.M.F. induced in the single coil of wire of the ideal alternator considered in § 57, and the desired result is obtained by rotating an armature consisting of *a large number of coils* at a *rapid rate* in an *intense magnetic field.* The magnetic flux which passes through the armature coils is concentrated by using a soft iron core upon which the insulated wire is wound. In many respects the armature is the vital part of a generator, and it is necessary at this point to determine the relationship between the E.M.F. induced, and the following factors upon which its magnitude depends, i. e. number of external armature conductors, the magnetic flux, and the speed:

Let N = total magnetic flux passing through the armature.

$\mathfrak{C}$ = total number of external wires connected *in series* on armature.

then if n = number of revolutions per minute,

$$(\text{E.M.F. in volts}) = \text{time-rate of change in flux linked with the armature coils} \div 10^8$$

$$= \frac{\text{flux (C.G.S. lines) cut per second}}{10^8}$$

and one conductor cuts 2 N lines per revolution

and the armature makes $\frac{n}{60}$ revolutions per second

$$\therefore \quad \text{E.M.F. for each conductor} = 2\,\mathsf{N} \times \frac{n}{60} \times \frac{1}{10^8} \text{ volts.}$$

Now in ordinary bipolar machines in which two brushes are used, the armature coils are divided into two equal portions, which are connected in parallel by the brushes. And, according to the law of the divided circuit, the E.M.F. of the combination is the same as

that induced in each half of the armature, therefore number of external conductors in each portion

$$= \frac{\mathfrak{C}}{2}$$

and $\qquad \frac{\mathfrak{C}}{2}$ conductors cut $2\,\mathsf{N} \times \frac{n}{60} \times \frac{\mathfrak{C}}{2}$ lines per second

and the E.M.F. induced in an armature is

$$\text{E (volts)} = \frac{\mathsf{N}\,\mathfrak{C}\,n}{60 \times 10^8}$$

This is the fundamental formula in dynamo design, and may be applied to the various types of armatures provided suitable constants be introduced. Thus, for multi-polar machines, with p pairs of poles, the E.M.F. is p times the value given above, if N be the magnetic flux for each pair of poles and the armature be series-wound. Thus

$$\text{E (volts)} = \frac{p\,\mathsf{N}\,\mathfrak{C}\,n}{60 \times 10^8}$$

Since the iron used for armature cores is generally very soft and of the best quality, the induction B_a (lines per square centimetre) is usually very high. For rings it varies from 15000 to 20000 lines per sq. cm. of actual iron, whilst for drum armatures in which the radial depth of iron is greater, the induction varies from 12000 to 15000 lines.

The speed varies considerably; in practice a peripheral speed of 2700 to 3000 feet per minute is common, and for large machines this is often exceeded.

For multipolar machines, with p pairs of poles, there are usually p pairs of brushes.

Worked Examples. (1) A closed coil of wire of one turn is moved through a magnetic field so that the number of lines embraced diminish from 750000 to zero in $\frac{1}{40}$th of a second. Determine the E.M.F. induced in volts.

Rate of change = 750000×40 lines per sec.

$\therefore$ induced E.M.F. = 30000000 absolute units

$$= \frac{3 \times 10^7}{10^8} = 0{\cdot}3 \text{ volt.}$$

(2) A coil of 100 turns of wire forming a loop of 25 square centimetres rotates in a uniform magnetic field, the intensity of which is 400 lines per square centimetre. At a certain instant the plane of the coil is perpendicular to the magnetic field, and is filled with lines of force; one-hundredth of a second later the plane of the coil is parallel to the lines of force, and is emptied of flux. What is the E.M.F. induced?

Change in the linkages per second

$$= \frac{n \,.\, \mathsf{N}}{t}$$

$$= \frac{100 \times (25 \times 400)}{\frac{1}{100}}$$

$$= 100000000$$

$$\therefore \quad \text{E.M.F.} = 100000000 \text{ absol. units}$$

$$= 1 \text{ volt.}$$

(3) What E.M.F. will be generated in a Gramme armature, revolving at 800 revolutions per minute, in a four-pole machine? There are 312 coils of wire on the armature connected in series, and the magnetic flux for each pair of poles is 3000000 C.G.S. lines.

Since the machine is multipolar

$$\text{E} = p . \frac{\mathsf{N}\, \mathcal{T} n}{10^8 \times 60}$$

where p = number of pairs of poles.

$$\therefore \quad \text{E} = \frac{2 \times 3000000 \times 312 \times 800}{10^8 \times 60}$$

$$= 31{\cdot}2 \times 8 = 249{\cdot}6 \text{ volts.}$$

(4) A Gramme armature, intended for a 2-pole field, has 120 turns of conductor wound upon it. Give the total flux required through the armature core if an electromotive force of 100 volts is to be induced at a speed of 1000 revolutions per minute. (C. and G. 94.)

Since $$\text{E (volts)} = \frac{\mathsf{N}\, \mathcal{T} n}{10^8 \times 60}$$

$$\therefore \quad 100 = \frac{\mathsf{N} \times 120 \times 1000}{10^8 \times 60}$$

$$= \frac{\mathsf{N} \times 2}{10^5}$$

$$\therefore \quad \mathsf{N} = \frac{10^5 \times 100}{2} = 5000000 \text{ (total flux).}$$

(5) Given a ring armature 9·25 inches long, 10 inches diameter, 1 inch radial thickness of iron, what wire would you use to give 110 volts, and what current would be suitable? Speed, 1000 revolutions per minute. (C. and G. 95.)

(*a*) *Determination of magnetic flux* (N) *through armature core.* Assuming that the armature is built up of the very best soft iron discs, the induction, B, may be taken as 20000, and also assuming the total thickness of paper insulation between the discs to be $\frac{1}{4}$ inch, the cross-sectional area of the armature core

$$= 2 \times 1 \times 9 = 18 \text{ sq. inches}$$

$$= 18 \times 6{\cdot}45 \text{ sq. cm.}$$

$$= 116 \text{ sq. cm.}$$

since $$1 \text{ sq. inch} = 6{\cdot}45 \text{ sq. cm.}$$

$\therefore$ the total magnetic flux, N, through the core

$$= 116 \times 20000 = 2320000 \text{ C.G.S. lines.}$$

(*b*) *Determination of number of armature conductors,* $\mathfrak{C}$.

Since
$$E\ (\text{volts}) = \frac{N\,\mathfrak{C}n}{10^8 \times 60}$$

$$\therefore \quad 110 = \frac{2320000 \times \mathfrak{C} \times 1000}{10^8 \times 60}$$

and
$$\mathfrak{C} = \frac{110 \times 600}{232} = 284$$

(*c*) *Determination of size of wire.*

The circumference of armature core

$$= 10 \times 3{\cdot}1416 = 31{\cdot}416 \text{ inches,}$$

and assuming that the conductors are arranged in one layer,

$$\text{the diameter of the insulated wire} = \frac{31{\cdot}416}{284}$$
$$= {\cdot}1106 \text{ inch.}$$

Now according to Professor S. P. Thompson, D.Sc., the usual insulation for round wires of a greater diameter than No. 16 S.W.G. is a double cotton covering, which increases the diameter by amounts varying from 10 to 20 mils, but which usually averages 14 mils. And from inspection of a table of wire gauges, the diameter of a No. 16 S.W.G. wire is ·064 inch, consequently, if ·014 inch be allowed for insulation

$$\text{the diameter of the bare wire} = {\cdot}1106 - {\cdot}014$$
$$= {\cdot}0966 \text{ inch}$$

which is approximately the diameter of a No. 13 B.W.G. wire.

(*d*) *Determination of permissible current.*

The area of the wire, ·0966 inch in diameter

$$= 0{\cdot}00733 \text{ sq. inch.}$$

And since each armature conductor carries only half the armature current, and as it is usual to allow 2000 amperes to the square inch with armature conductors, the permissible current

$$= 2 \times 2000 \times {\cdot}0073 \text{ amperes}$$
$$= 29{\cdot}2, \text{ or, say } 30 \text{ amperes.}$$

EXERCISES VII A.

(1) A metal disc is making 2400 r.p.m. in a uniform field, whose strength perpendicular to the disc is 10000 C.G.S. units. Determine the magnitude of the electrical pressure produced if the diameter of the disc is 20 cms.

(2) Two parallel wires lying N and S are connected by two wires laid perpendicularly across them. The north one moves northwards with a velocity of one kilometre per second. If the wires be one metre apart determine the electrical pressure generated, taking the earth's vertical force = 0·47 C.G.S. unit.

(3) A river 100 metres broad is flowing along an insulating bed at the rate of two metres per second. Calculate the P.D. between the

water and the two sides being given that $H = 0{\cdot}17$ and that the magnetic dip is 60°.

(4) Determine the E.M.F. generated in an armature, the diameter of which is $1\frac{1}{24}$ feet, the surface velocity of which is 2200 feet per minute. The number of conductors is 176, and it revolves in a field, the total magnetic flux of which is 7812500.

(5) A drum armature in a 2-pole field contains 150 external conductors, and runs at 550 revolutions per minute. Find the total flux passing through the armature, which is required to produce an electromotive force of 115 volts on open circuit. (C. and G. 93.)

(6) A wire is bent so as to form a complete rectangular coil 20 cms. by 10 cms., and fixed so as to make 600 r.p.m. in a field of 12000 units strength. Determine the value of the E.M.F. generated in the wire at the instant that it is a max. and also at an instant $\frac{1}{80}$ second later.

(7) Determine the areas of cross section of the armature cores of two machines, one with a ring-wound armature ($B = 16000$), and the other with a drum-wound armature ($B = 12000$), if each armature has 256 active conductors, and generates 125 volts at 1260 revolutions per minute.

(8) Given that a drum-wound armature with 240 conductors has a cross-sectional area of core of 250 sq. cms., and an armature resistance of 0·025 ohm. Determine the E.M.F. generated at a speed of 20 revs. per second if $B = 10000$.

(9) If the armature (No. 8) is required to give a terminal P.D. of 120 volts when a current of 120 amperes traverses the armature, determine the increase in speed, assuming that $B = 10000$.

(10) A dynamo has 90 conductors on the armature, the reluctance of the magnetic circuit is 0·002 C.G.S. units, and there are 1500 turns of wire carrying 8 amperes on the field magnets. If the leakage coefficient is 1·3, determine the speed at which it must be driven to give 100 volts on open circuit.

(11) A dynamo with 130 armature conductors and running at 720 revolutions per minute has 200 amperes passing through the armature, which has 0·025 ohm resistance. The terminal P.D. of the machine is found to be 105 volts. Determine the magnetic flux passing through the armature.

(12) Determine the flux density for a drum armature 25 cms. diameter with a 6 cm. hole at the centre, if the core consists of 600 discs 0·5 mm. thick. There are 200 conductors on the armature and 200 volts are produced when it runs at 1000 revolutions per minute.

(13) An armature core of a bipolar machine is 25 cms. long and

10 cms. deep. There are 48 sections with 5 coils in each on the armature, and the armature resistance is 0·25 ohm. Determine the terminal voltage when it runs at 1500 r.p.m. and a current of 150 amperes traverses the armature, if $B = 12000$.

(14) A drum-wound armature is built up of 500 core discs 0·025 inch thick of charcoal iron, and of 8 inches external diameter and 2 inches internal diameter. The number of armature conductors is 200, and the E.M.F. generated is 180 volts when running at 900 revolutions per minute. Determine the flux density.

(15) Calculate the resistance of a Gramme armature wound with 144 turns of rectangular wire $0{\cdot}2 \times 0{\cdot}21$ inch, length of armature core without insulation 12 inches, radial depth 2·5 inches. The resistance of 100 yards of copper rod, one square inch in cross section, is 0·0025 ohm. (C and G.)

(16) A two-pole dynamo is required to give a net output of 10 E.H.P. at 200 volts running at 1000 r.p.m. Given the following particulars, estimate the cross-sectional area of the field magnet limbs and the number of turns of wire on the magnet coils.

Number of conductors round the armature = 350
Air gap = 1 cm. Polar angle = 135°
Length of armature core = its diameter = 28 cms.
Coefficient of leakage 1·3
Average total length of magnetic circuit = 140 cms.
Average flux density in the iron portion = 15000 per sq. cm.
Armature resistance = 0·15 ohm.
Average permeability of iron in magnetic circuit = 500
Field current = $3\frac{1}{2}$ per cent. of output.

(17) Find the ampere-turns for the shunt winding of a dynamo of the following dimensions:—

Type. Single horse-shoe 'Overtype.'

Output. 200 amperes at 110 volts at 800 r.p.m.

Armature drum-wound 120 conductors, 60 commutator bars, resistance = 0·025 ohm.

Armature Core. Pure iron, external diameter $11\frac{1}{16}$ inches; section of iron in core = 60 square inches; mean path through core = 7 inches; length of core = 16 inches.

Pole-pieces and *Field magnet cores* formed of pure iron. Diameter of bore = 12 inches; polar angle = 120°; mean length of path = 36 inches; mean area = 95 square inches.

Yoke formed of cast steel; mean length of path = 12 inches; mean area = 150 square inches. Leakage coefficient at full load 1·45.

Use table, p. 200, columns *I* and *V*.

(18) You are given a 2-kilowatt shunt dynamo of the following

construction. Field turns = 2600; armature conductors = 400; output = 30 amperes at 65 volts. How would you rewind it to give an output of 5 amperes at 40 volts, with the same efficiency as before?

(19) You have a cylindrical armature core 12 inches diameter, 15 inches long, and two-pole field, bore of pole field 13·4 inches, polar angle 112°. The armature has 150 external conductors, and carries a current of 250 amperes. Determine the following: (1) total flux of lines to produce an armature E.M.F. of 115 volts at 550 revolutions per minute; (2) exciting power required for air-space; (3) average induction in air-space; (4) induction under the polar edges at full load. (C. and G.)

(20) Determine the winding of a Gramme armature, the core of which has a length of 12 inches (not including the thickness of the insulation), a diameter of 7 inches, and a radial depth of iron of 2 inches. The bore of the pole-pieces is not to exceed 13 inches. The dynamo is to produce 100 volts at 1000 revolutions per minute, and the largest current permissible so as not to raise the temperature of the armature more than 75° F. above that of the air. Assume such an induction as you think desirable. Calculate the resistance of the armature cold and hot. (C. and G.)

(21) A firm has in stock a field magnet wound with 1540 turns on each of its two limbs, and having a total resistance of 14 ohms warm. The length of the mean line is 80 centimetres, and the polar bore $14\frac{1}{2}$ inches in diameter. Consider whether it is possible to construct an armature for this field to give 400 amperes at 110 volts at a speed of 540 r.p.m. If it be possible, give the diameter of the smooth core you would use, the number and size of the armature bars, and state whether the machine would have any faults which would not have existed had you designed an entirely new machine. The length of the magnet parallel with the shaft is 20 inches and its smallest width 9 inches. The materials used in the armature and field cores have the following qualities:—

For a density	= 11000	C.G.S. lines in the armature	H = 5·6
,, ,,	= 12000	,, ,, ,, ,,	H = 7·6
,, ,,	= 13000	,, ,, ,, ,,	H = 11·3
,, ,,	= 14000	,, ,, ,, ,,	H = 16·9
,, ,,	= 12000	,, ,, in the field magnets	H = 10
,, ,,	= 13000	,, ,, ,, ,,	H = 14
,, ,,	= 14000	,, ,, ,, ,,	H = 18

(C. and G.)

Section IV. Drag on Armature Conductors.

§ **60. Simple Cases.** Whenever a wire carrying a continuous current is placed in a magnetic field so that the current is in a direction at right angles to the lines of force of the field, the conductor experiences a force tending to move it perpendicularly to itself, and also to the direction of the lines of force of the field. This part of the subject will receive full treatment in Chapter IX, but we may state here that this force is proportional to the strength of the current, to the intensity of the magnetic field, and to the length of the conductor; or algebraically

$$F = K \times H \times C \times l$$

where H = intensity of magnetic field
C = the current
l = the length of the conductor
and $K \doteq$ a constant depending on the system of units used.

If F be in dynes, H in C.G.S. lines per sq. cm., C current in absolute units, l length in cm., then $K = 1$, and

$$F = H \times C \times l \text{ dynes.}$$

If C be given in amperes, then $K = \frac{1}{10}$

and
$$F = \frac{1}{10} H\, C\, l \text{ dynes.}$$

As an example, let l be one foot, therefore l equals $12 \times 2{\cdot}54$ cm., since 1 inch = 2·54 cm., and let $H = 10000$, $C = 500$ amperes, then substituting the values given

$$F = \frac{1}{10} \times 10000 \times 500 \times 12 \times 2{\cdot}54 \text{ dynes per foot-length}$$
$$= 15240000 \text{ dynes}$$
$$= \frac{15240000}{981} \text{ grammes-weight.}$$

Since a force of 981 dynes = weight of one gramme.
And since one pound = 453·6 grammes,

$$F = \frac{15240000}{981 \times 453{\cdot}6} \text{ pounds-weight}$$
$$= 34{\cdot}25 \text{ pounds.}$$

It is important to notice that this force, due to the interaction between a current and a magnetic field, has to be overcome whenever a current passes through the armature of a dynamo; for by Lenz's Law we know that the direction of the induced E.M.F. in the

armature of a dynamo (and consequently the direction of the resulting current) is such that it tends to stop the motion producing the E.M.F. This resistance to motion is termed *a drag on armature conductors;* and which with dynamos necessitates a driver or prime mover, is the force which causes the armature to rotate in motors.

Since the average value of H in the air gaps of modern dynamos and motors may be taken as 5000 lines per sq. cm., it is an easy matter to determine the pull per foot-length per ampere on armature conductors in the strong parts of the interpolar space.

Thus
$$P = \tfrac{1}{10} \times 5000 \times 1 \times 12 \times 2{\cdot}54 \text{ dynes}$$
$$= \frac{500 \times 30{\cdot}48}{981 \times 453{\cdot}6} \text{ pounds-weight}$$
$$= {\cdot}0345 \text{ pounds-weight.}$$

When considering the total pull on the armature we must remember there is a want of uniformity in the intensity of the field between the pole pieces, and for this reason it is usual to take the conductors subtended by the poles as being actively concerned in this drag on the armature as a whole. Thus if θ be the angle of polar embrace, the number of conductors acted upon is

$$\mathfrak{C}_1 = \mathfrak{C} \times \frac{2\,\theta}{360}$$

where $\mathfrak{C}$ is the total number of conductors on the armature. Usually θ varies from 120° to 140°. If the full armature current is given, we must also remember that only half of this flows through each conductor, consequently for bipolar machines

$$P = \frac{1}{10}\,\mathsf{H} \times \frac{C_a}{2} \times l_a \times \frac{\theta}{180}\,\mathfrak{C} \text{ dynes}$$

if l_a be given in cms.

Worked Examples. (1) An armature 20 inches in length and 12 inches in diameter consists of 120 conductors. The angle of embrace is 120°, and 400 amperes is produced. The intensity of the field is 5000 lines. Determine the drag on the armature.

$$l_a = 20 \times 2{\cdot}54 \text{ cms.}$$

and
$$P = \frac{1}{10} \times 5000 \times \frac{400}{2} \times 20 \times 2{\cdot}54 \times \frac{2 \times 120}{360} \times 120 \text{ dynes}$$
$$= 500 \times 200 \times 50{\cdot}8 \times 80 \text{ dynes}$$
$$= \frac{100000 \times 4064}{981 \times 453{\cdot}6} \text{ pounds-weight}$$
$$= 913{\cdot}4 \text{ pounds-weight.}$$

(2) Find the torque required and the energy expended in producing the 400 amperes in the above example, given that the armature makes 500 revolutions per minute.

The torque is that turning moment experienced at the circumference of the armature,

$\therefore$ force acting at a radius of ·5 feet produces a

$$\text{torque} = 913{\cdot}4 \times {\cdot}5 = 456{\cdot}7 \text{ pound-feet.}$$

Now work expended per revolution

$$= 913{\cdot}4 \times \pi \text{ foot pounds,}$$

$$\text{and the energy required} = \frac{913{\cdot}4 \times \pi \times 500}{33000} \text{ H.P.}$$

$$= 43\tfrac{1}{2} \text{ H.P. (approx.).}$$

EXERCISES VII B.

(1) A wire carries 100 amperes, and is placed in a field of 2000 C.G.S. units strength. Determine the force experienced by a length of one metre.

(2) In the equation $F = K \times H \times C \times l$ prove that $K = 57 \times 10^{-8}$, if F be measured in pounds, H in C.G.S. units, C in amperes, and l in inches.

(3) Determine K in the equation, $F = K \times H \times C \times l$, if F be measured in pounds, H in Kapp lines per sq. inch, C in amperes, and l in inches.

Note. One Kapp line = 6000 C.G.S. lines.

(4) Determine the drag on an armature 20 inches long, in which 326 amperes is produced, if H per sq. inch is 43300, $\theta = 120°$ and the number of conductors is 240.

(5) A straight wire carrying a continuous current of 500 amperes lies at right angles to the magnetic flux in a field, the intensity of which is 10000 lines per square centimetre. State the force in pounds per foot of the wire tending to produce lateral displacement. (C and G. 94.)

(6) An insulated copper bar, forming part of the winding of a drum armature, carries a current of 50 amperes, and is in a magnetic field of 3500 units strength for a length of 25 cms. Determine the lateral force on the bar in pounds.

CHAPTER VIII

TYPES OF DYNAMOS

Section I. Series-wound Dynamos

§ 61. Methods of Exciting Field Magnets. Series-wound Dynamos. As we have shown, the electromotive-force induced in the armature conductors is proportional to the time-rate of change in the magnetic flux looped with the armature conductors; it is, therefore, of paramount importance to obtain the magnetic field, in which the armature rotates, as intense as possible. In the modern generators electromagnets with large pole-pieces are used to provide the intense magnetic fields required. Field magnets, however, are the result of a development which is another example of the 'survival of the fittest' law, and, although it would be interesting to trace the growth of the dynamo from the disc machine of Faraday to the modern generator, space will not permit more than the enumeration of the more important methods of obtaining uniform and strong magnetic fields so that we may classify the various types of machines. In the earliest machines compound permanent steel magnets were used, to which the name *magneto-electric machines* has been given. As the compound permanent magnets were costly to build up in comparison with the relatively small outputs obtained, and also gradually became demagnetized, the magneto-electric machine soon gave place to a machine with electromagnets. At this stage the electromagnets were supplied with current from a separate and external source (battery or magneto-electric machine), hence these are known as *separately-excited* machines. The most interesting principle which has been introduced, however, is that of *self-excitation*, which was devised by S. A. Varley in 1866, and soon after independently introduced by Siemens and Wheatstone; self-excitation is the principle utilized in continuous-current generators of the present time. The action of self-excited machines is due to the residual magnetism retained by the magnets after being initially magnetized, and to the method of taking the whole or part of the armature current through the field magnet coils. Self-excitation is therefore an example of the compound interest principle, since the residual magnetism permits a small electromotive-force to be generated in the armature as soon as it revolves, which in its turn sends a small

current through the field magnet coils, and so increases the field magnetism, in consequence of which more pressure is generated, and this goes on until the required excitation is reached, when the machine will give its normal output. As we have just inferred, there are several distinct types of self-excited machines, according to the relation of the excitation current to the armature (either whole or part) and external currents, and we may distinguish the various types of self-excited generators by dividing them into (1) *series-wound* dynamos, (2) *shunt-wound* dynamos, and (3) *compound-wound* dynamos.

We thus have the following classification of dynamos, according to the manner in which the magnetic fields are produced:

(1) Magneto-electric machines.
(2) Separately excited machines.
(3) Self-excited machines, divided into
    (1) Series-wound dynamos.
    (2) Shunt-wound dynamos.
    (3) Compound-wound dynamos.

The series-wound dynamo (Fig. 52) has its field magnets excited

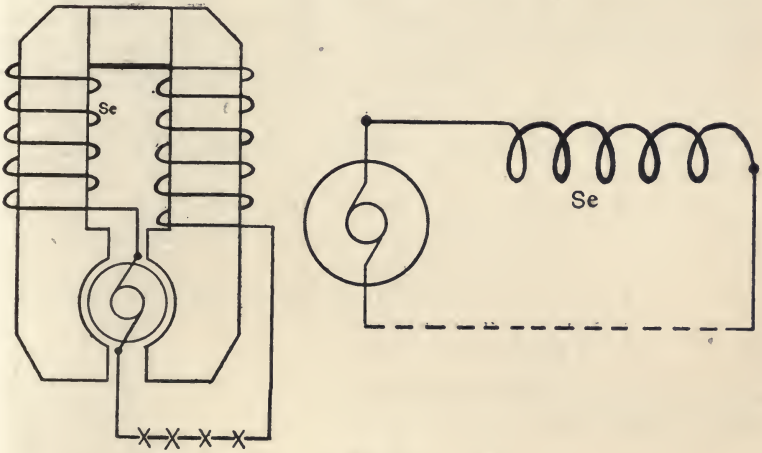


Fig. 52.

by the full armature current which is taken directly from the armature around the field magnets by comparatively few turns of thick wire, cable, or ribbon, and thence to the external circuit back to the armature. The complete circuit is therefore composed of three parts—the armature coils, the field magnet coils, and the external portion, lamps, &c.—all connected in series with one another. This is manifestly the simplest and most primitive method of utilizing the current produced by a generator to excite its own field magnets. Further-

more, the series-wound dynamo is particularly well adapted for series arc-lighting, which, with the exception of electro-plating, was the first application of electricity to industrial uses on a commercial scale. In the early days the number of arc lamps required was comparatively small, and the primitive method of connecting them in series was found convenient, and was universally adopted. Series-wound dynamos, therefore, may be considered as the first of the really satisfactory commercial machines.

The series-wound dynamo is practically a constant current generator, but it does not regulate itself automatically to the work it has to do. Since arc lamps are usually connected in series it is obvious that the series dynamo designed for series arc lighting is a high tension generator, and considerable difficulty is experienced in insulating the commutator segments.

The main objection to the series-wound dynamo is that it does not start working below a certain speed, or if the external resistance is too great, as it then fails to excite itself. It furthermore possesses the disadvantage of being liable to reverse its polarity should the speed fall when there is any source of counter E.M.F. in the circuit. Consequently it is altogether unsuitable for charging accumulators, or for electro-metallurgical work generally. It is manifest also that immediately the circuit is broken at any point, the current ceases and the excitation is removed and the machine becomes inactive.

To determine the various quantitative relationships and efficiency of a series dynamo, let

Ea E = total E.M.F. generated in the armature,
e = terminal difference of potential or voltage,
r_a = resistance of armature,
r_{se} = resistance of field coil, in series with the armature,
R = resistance of external circuit,
I C = current,
η_e = electrical efficiency,

and by applying Ohm's Law we have:

$$C = \frac{E}{R + r_a + r_{se}} = \frac{e}{R}$$

drop in volts in armature, $e_a = C\,r_a$

„ „ series coils, $e_{se} = C\,r_{se}$

$\therefore$ lost volts, $v = e_a + e_{se} = C\,(r_a + r_{se})$

and $e = E - v = E - C\,(r_a + r_{se})$

$$\therefore\quad E = e + C\,(r_a + r_{se})$$
$$= e\left\{1 + \frac{(r_a + r_{se})}{R}\right\}$$

$$\left.\begin{aligned}\text{Watts lost in armature} &= C\,e_a = C^2 r_a \\ \text{,, \quad ,, \quad series coils} &= C\,e_{se} = C^2 r_{se} \\ \text{,, \quad used externally} &= C\,e = C^2 R\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{Also, } \eta_e = \frac{\text{energy utilized}}{\text{energy developed}} &= \frac{e\,C}{E\,C} = \frac{e}{E} \\ &= \frac{C^2 R}{C^2 R + C^2 r_a + C^2 r_{se}} \\ &= \frac{R}{R + r_a + r_{se}}\end{aligned}\right\}$$

From the consideration of the construction of the series dynamo and the arrangement of the circuit it is obvious that if the external resistance be diminished the current increases; but an increase in the current strength means increased excitation, and consequently a higher E.M.F., when, really, less is required. On the other hand, an increase in the resistance of the external circuit produces a diminution of the current, resulting in a reduced magnetic field, and in neither case is the series dynamo capable of automatically regulating itself to its load.

It is also manifest, from the above equations, that to make the efficiency a maximum $(r_a + r_{se})$ must be small; and according to Lord Kelvin r_a should be slightly greater than r_{se}.

Worked Examples. (1) A seven-kilowatt arc-lighter generates an E.M.F. of 780 volts at a certain speed, and supplies a current of 10 amperes for lighting 13 fifty-volt arc lamps, connected in series. $r_a = 3{\cdot}448$ ohms, and $r_{se} = 4{\cdot}541$ ohms. Determine the volts lost in the armature, the terminal voltage of the machine, and the length of the cable used, given that the resistance of the cable is 4·3 ohms for 1000 yards.

Solution: (*a*) *Determination of volts lost in armature.*

$$\begin{aligned}\text{Lost volts in armature} &= C \times r_a \\ &= 10 \times 3{\cdot}448 = 34{\cdot}48 \text{ volts.}\end{aligned}$$

(*b*) *Determination of terminal voltage.*

In addition to the volts lost in the armature there is the drop in the volts in the field magnet coils

$$\begin{aligned}&= C \times r_{se} = 10 \times 4{\cdot}541 \\ &= 45{\cdot}41 \text{ volts}\end{aligned}$$

$$\begin{aligned}\therefore \quad \text{total volts lost internally} &= 34{\cdot}38 + 45{\cdot}41 \\ &= 79{\cdot}89 \text{ volts} \\ \therefore \quad e = \text{terminal voltage} &= 780 - 79{\cdot}89 \\ &= 700 \text{ volts.}\end{aligned}$$

(*c*) *Determination of the length of cable used.*

$$\text{Resistance of each lamp} = \frac{50 \text{ volts}}{10 \text{ amperes}} = 5 \text{ ohms}$$

and $\quad$ total lamp resistance $= 13 \times 5 = 65$ ohms.

Let R = total resistance of cable

then since $$C = \frac{E}{R}$$

$$10 = \frac{700}{R+65}$$

and $R+65 = 70 \qquad \therefore\ R = 5$ ohms.

But 4·3 ohms is the resistance for 1000 yards,

$\therefore$ 4·3 ohms : 5 ohms : : 1000 yards : x

and $$x = \frac{5 \times 1000}{4{\cdot}3} = 1162{\cdot}8 \text{ yards};$$

$\therefore$ length of cable = 1162·8 yards.

Note. For series-wound machines

$$\left.\begin{aligned} E &= C\,(R+r_a+r_{se}) \\ \text{and}\quad e &= CR \end{aligned}\right\} \quad \therefore \quad \frac{e}{E} = \frac{R}{R+r_a+r_{se}}$$

and
$$\begin{aligned} e &= \frac{R}{R+r_s+r_{se}} \times E \\ &= \frac{70}{70+3{\cdot}448+4{\cdot}541} \times 780 \\ &= \frac{70}{77{\cdot}989} \times 780 \\ &= 70 \times 10 = 700 \text{ volts.} \end{aligned}$$

EXERCISES VIII A.

(1) A series-wound dynamo giving 12·5 amperes at a terminal voltage of 3000 volts receives 62·5 h.p. at the pulley. What energy is absorbed by the machine, and what is its commercial efficiency?

(2) A series-wound dynamo receives 7·5 h.p. at its pulley and gives a certain current at a terminal voltage of 275 volts. If the commercial efficiency is 81·59, what is the current?

(3) The terminal P.D. of a series machine is 1200 volts, and the current supplied is 12·5 amperes. If the internal resistance is 12 ohms, how much energy is absorbed by the machine? What E.M.F. is generated by the machine, and what is its electrical efficiency?

(4) An engine delivers 32 h.p. at the pulley of a series-wound generator giving 100 amperes. The internal resistance is 0·24 ohm, and the resistance of the external circuit is 1·8 ohms. Determine the energy lost internally, the E.M.F. generated, the electrical and commercial efficiencies.

(5) A series-wound dynamo gives 20 amperes at a terminal voltage of 800 volts. The armature resistance, r_a, is 0·5 ohm, the field winding resistance, r_{se}, is 1·3 ohms. If the commercial efficiency

is 80 per cent., determine the E.M.F. generated and the power supplied at the pulley of the machine to drive it.

(6) If $r_a : r_{se} = 0{\cdot}45 : 1$ and $r_a = 0{\cdot}5$ ohm for a series-wound dynamo, determine (1) the electrical efficiency, (2) the resistance of the external circuit if the E.M.F. generated is 830 volts and the current 20 amperes.

(7) Determine the energy dissipated in the armature and in the field magnet winding of the machine considered in No. 6.

(8) The armature resistance of a series-wound generator is 0·25 ohm, that of the field magnet winding 0·75 ohm, and that of the external circuit 12 ohms. If the current is 15 amperes, determine the E.M.F. generated, the volts lost internally, and the output.

(9) A series arc-lighting machine gives 10 amperes at a terminal P.D. of 1200 volts. The armature resistance is 7·2 ohms, and the resistance of the field magnet coils is 15·75 ohms. If the commercial efficiency is 78·5 per cent., determine the number of watts wasted as friction and hysteresis.

(10) Determine the E.M.F. generated in the above case (No. 9), and the ratio of the energy wasted in the armature and field magnet coils to that used externally.

Section II. Shunt-wound Machines.

§ **62. Shunt-wound Dynamos.** Since glow lamps are, in practice, connected in parallel between mains, it is essential to maintain these mains at a constant difference of potential, and from what was said in the last section it is clear that the series-wound dynamo—being a constant current generator, and unable to give a constant voltage with varying loads—is not adapted for incandescent lighting. To avoid fluctuation of the voltage with varying loads the shunt-wound dynamo was introduced. This machine is an example of self-excited generators in which only part of the armature current is used for purposes of excitation, the field magnet coils consisting of many turns of fine wire connected in parallel—i.e. as a shunt—to the external or main circuit, as shown in Fig. 53; it is for this reason that they are termed *shunt-wound* dynamos. There are, consequently, points of difference between shunt and series dynamos. In the first place, the shunt-wound dynamo remains excited with the external circuit open provided the brushes are in use and the armature rotates at its proper speed. Again, since the field magnet coils consist of a large number of turns of fine wire the resistance of the shunt coils is great, and only a small fraction of the current generated passes through them. It is also easy to see that with this

arrangement the direction of the current in the field magnet coils remains the same, even if a counter electromotive force exist in the external circuit and becomes greater than the difference of potential generated in the armature. In other words, the possibility of reversing the polarity of the shunt dynamo (as is the case with the series machine) does not exist. It is, therefore, particularly adapted for electrochemical and metallurgical work; it is essentially the most suitable generator for charging accumulators. Speaking generally, the essential feature of the shunt-wound dynamo is that of giving a *fairly* constant voltage with varying load, and the student will have no difficulty in grasping the theory of the working of

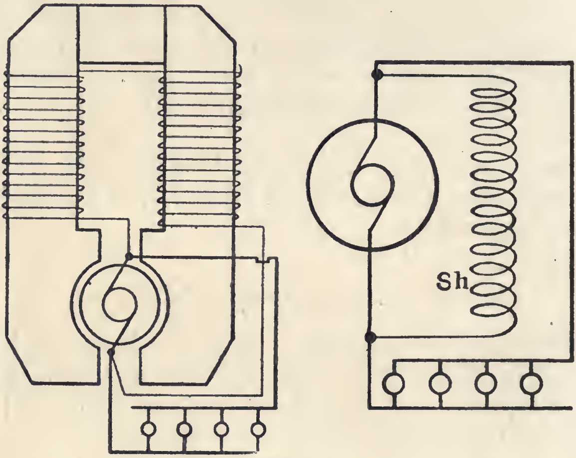


FIG. 53.

a shunt dynamo after he has proved and understands the following relationships which are obtained by applying Ohm's Law. In addition to the symbols used in Section I, let

$$C_a = \text{current in the armature}$$
$$r_{sh} = \text{resistance of shunt coils}$$
$$C_{sh} = \text{current in the shunt coils.}$$

Then

$$\left.\begin{aligned} C_a &= C + C_{sh} \\ C_{sh} &= \frac{e}{r_{sh}} = \frac{R}{r_{sh} + R} \times C_a \\ \frac{C_{sh}}{C} &= \frac{R}{r_{sh}} \end{aligned}\right\}$$

Also

$$\text{Total resistance} = r_a + \frac{R \times r_{sh}}{R + r_{sh}} \Big\}$$

$$\text{Current in the armature, } C_a = \frac{E}{r_a + \dfrac{R\,r_{sh}}{R + r_{sh}}}$$

$$= \frac{e}{R} + \frac{e}{r_{sh}}$$

$$= e\left(\frac{R + r_{sh}}{R\,r_{sh}}\right)$$

$$= \frac{e}{R}\left\{1 + \frac{R}{r_{sh}}\right\}$$

Volts lost in the armature $= v = C_a r_a = r_a\,(C + C_{sh})$

and
$$E = e + v = e + C_a r_a$$

$$= C_a\left\{r_a + \frac{R\,r_{sh}}{R + r_{sh}}\right\}$$

$$= e\left\{1 + \frac{r_a}{R} + \frac{r_a}{r_{sh}}\right\}$$

$$= e r_a\left\{\frac{1}{r_a} + \frac{1}{r_{sh}} + \frac{1}{R}\right\}$$

terminal P. D. or voltage $= e = C\,R = C_{sh}\,r_{sh}$

Energy expended in the armature $= C_a^2\,r_a$

" " " shunt coils $= C_{sh}^2\,r_{sh}$

" " " external circuit $= C^2\,R$ (useful work)

$\therefore$ the electrical efficiency η

$$= \frac{C^2 R}{C^2 R + C_a^2 r_a + C_{sh}^2 r_{sh}}$$

$$= \frac{1}{1 + \dfrac{r_a}{R} + 2\dfrac{r_a}{r_{sh}} + \dfrac{R}{r_{sh}}\left(1 + \dfrac{r_a}{r_{sh}}\right)}$$

Worked Examples. (1) A 19-unit shunt dynamo has an electrical efficiency of 95 per cent. If the terminal P.D. is 100 volts, determine

(1) the total current in the armature at full load,
(2) the resistance of the armature,
(3) the resistance of the shunt coils;

assuming an equal loss in the armature and shunt coil.

19 units or 19000 watts = 95 per cent. of energy generated

$\therefore$ energy lost in armature and shunt $= \frac{5}{95}$ of 19000

$= 1000$ watts

$\therefore$ energy lost in shunt coils $= 500$ watts

$= C_{sh}\,e$

$$\therefore \quad C_{sh} \times 100 = 500 \text{ watts}$$

and $$C_{sh} = 5 \text{ amperes}$$

$$\therefore \quad \text{armature current } C_a = C + C_{sh}$$
$$= \frac{19000}{100} + 5$$
$$= 195 \text{ amperes}$$

$$\text{armature resistance, } r_a = \frac{500}{C_a^2}$$
$$= \frac{500}{(195)^2} = 0{\cdot}0131 \text{ ohm}$$

$$\text{resistance of shunt coil, } r_{sh} = \frac{e}{C_{sh}} = \frac{100}{5}$$
$$= 20 \text{ ohms.}$$

(2) A 21-unit shunt machine giving 200 amperes has 31 ohms resistance in the shunt coils. Determine

(*a*) the volts lost in the armature,

(*b*) the total current in the armature at full load,

(*c*) the resistance of the armature, and

(*d*) the ratio of the useful energy to the energy generated, given that the P.D. at the terminals is $\frac{21}{22}$nds of the E.M.F. generated.

(*a*) *Determination of lost volts.*

A 21-unit machine gives out 21×1000 watts

and $$21 \times 1000 = e \times C = e \times 200$$
$$\therefore \quad e = 105 \text{ volts}$$

and $$e = \frac{21}{22} E$$

$$\therefore \quad E = \frac{22}{21} \times 105 = 110 \text{ volts}$$

$$\therefore \quad \text{lost volts} = E - e = 110 - 105$$
$$= 5 \text{ volts.}$$

(*b*) *Determination of current in the armature.*

$$C_a = C + C_{sh}$$

and $$C_{sh} = \frac{e}{r_{sh}} = \frac{105}{31} = 3{\cdot}4 \text{ amperes.}$$

$$\therefore \quad C_a = C + C_{sh} = 200 + 3{\cdot}4 = 203{\cdot}4 \text{ amperes.}$$

(*c*) *Determination of armature resistance.*

$$r_a = \frac{E - e}{C_a}$$
$$= \frac{110 - 105}{203{\cdot}4} = \frac{5}{203{\cdot}4} = 0{\cdot}0245 \text{ ohm.}$$

(*d*) *Determination of ratio of useful energy to energy generated.*

$$\text{Useful watts} = C \times e = 200 \times 105.$$
$$= 21000 \text{ watts.}$$

$$\text{Total watts} = C \times e + C_{sh} \times e + C_a \times (E - e)$$
$$= 200 \times 105 + 3{\cdot}4 \times 105 + 203{\cdot}4 \times 5$$
$$= 21000 + 357 + 1017 = 22374 \text{ watts}$$

Note. Total watts $= C_a \times E = 203 \cdot 4 \times 110$
$= 22374$ watts

$$\therefore \quad \text{ratio} = \frac{21000}{22374} = \frac{0 \cdot 9386}{1} = 93 \cdot 86 \text{ per cent.}$$

EXERCISES VIII B.

(1) The armature resistance of a shunt-wound machine is 0·025 ohm, and the shunt resistance $33\frac{1}{3}$ ohms. If the armature current is 99·6 amperes, and the resistance of the external circuit 1·25 ohms, determine the output of the machine, the electrical efficiency, the terminal P.D., and the E.M.F. generated.

(2) A shunt-wound dynamo has an armature resistance of 0·021 ohm, and the resistance of the shunt winding is 70 ohms. If terminal voltage at full load is 110 volts, and the armature current $81\frac{4}{7}$ amperes, determine the output and the watts lost internally.

(3) The field magnet winding of a shunt machine has a resistance of 25 ohms and an armature resistance of 0·04 ohm. If the full load consist of 200 100-volt lamps connected in parallel, and each requires 0·6 ampere, determine the E.M.F. generated and the current in the armature, neglecting the drop in the mains.

(4) The load on a shunt machine consists of 250 glow lamps connected in parallel, each taking 0·5 ampere at 110 volts. If the loss in the mains is 4 per cent. of the energy utilized, and the electrical efficiency 90 per cent., determine the E.M.F. generated when the losses in the armature are equal to those in the shunt coils.

(5) Determine the resistance of the armature and the resistance of the shunt winding in the machine of (4).

(6) A shunt wound dynamo supplies 600 amperes at a terminal voltage of 110 volts; $r_a = 0 \cdot 005$ ohm and $r_{sh} = 40$ ohms. Determine the losses in the armature and in the field magnet winding, and the power required to drive the machine, if the frictional and other losses amount to 5 per cent. of that supplied to the external circuit.

(7) In a shunt dynamo $r_a = 0 \cdot 008$ ohm, $r_{sh} = 25$ ohms. If the output is 160 amperes at 65 volts, determine the energy generated, and the commercial efficiency if the mechanical energy spent in driving the machine is 16·75 h.p.

(8) In a shunt dynamo $r_a = 0 \cdot 013267$ ohm, $r_{sh} = 32$ ohms. If the exciting current is 3·5 amperes, and its electrical efficiency is 96 per cent. when $R = 0 \cdot 7$ ohm, determine: (*a*) the terminal voltage; (*b*) current in the armature; (*c*) the E.M.F. generated and the current supplied externally.

(9) A 20-unit machine is working with an electrical efficiency of 95 per cent. If the main current is 200 amperes, and $r_{sh} = 20$ ohms, determine

(*a*) the armature current at full load;
(*b*) resistance of the armature;
(*c*) the percentage loss in the armature.

(10) A 21-unit shunt machine supplies current to a circuit of 2·1 ohms resistance and $r_{sh} = 52{\cdot}5$ ohms. Determine

(*a*) E.M.F. generated;
(*b*) the total current in the armature at full load;
(*c*) the resistance of the armature;

given that the P.D. is $\frac{21}{22}$ of the E.M.F. generated.

(11) In a 33 kilowatt shunt-wound Edison-Hopkinson dynamo, particulars of the magnetic circuit of which were given on p. 212, $r_a = 0{\cdot}01$ ohm, $r_{sh} = 16{\cdot}93$ ohm, $C = 320$ amperes, and $e = 105$ volts when the machine runs at 750 revolutions per minute. Determine C_{sh}, C_a, lost volts in armature, E, watts wasted in the armature, watts wasted in the shunt coils, useful watts, and the total watts.

Section III. Compound-wound Dynamos.

§ 63.—Constant Potential Generators. Short-shunt and Long-shunt Compound-wound Dynamos.—The shunt-wound dynamo does not give that constant pressure at the terminals which is necessary for supplying incandescent lamps connected in parallel, since the drop in volts in the armature grows with the increasing current, as more lamps are used in the external circuit. It is obvious, therefore, that the difference of potential at the terminals falls and the excitation is weakened correspondingly. Furthermore, the armature reactions become more marked as the armature current increases, thus assisting in the tendency to diminish the electromotive-force generated. It will not be out of place here, however, to remark that in well-designed shunt machines with very small resistance in the armature there is a fair amount of *self-regulation* between certain limits. In order to produce that self-regulation which will give a high degree of constancy of potential, the field magnets of constant potential generators are usually wound with series coils in addition to the shunt coils; such a combination of windings gives a compound winding, and the machines are termed *compound-wound* dynamos.

In these generators the shunt coils are designed to provide the excitation required for the production of the requisite difference of potential on open circuit, whilst the series coils are added to compensate for the drop in volts when the machine is loaded. The

combination of windings, in fact, overcomes the back-excitation of the armature, makes good the lost volts at full load, and by maintaining constant excitation gives a constant potential with varying loads.

The shunt winding consists of many turns of fine wire as in the shunt machine, and the series winding consists of comparatively few turns of thick wire, and is traversed by the main current. There are two methods of connecting the shunt coils: (1) the shunt winding may be connected from brush to brush, i. e. as a shunt to the series winding and the external circuit, and (2) the shunt winding may be connected across the terminals of the machine, i. e. as a shunt to the external circuit. The former is known as the *short-shunt* or *ordinary compound winding*, and the latter as the *long-shunt compound winding*. The two types are shown diagrammatically in Figs. 54 and 55.

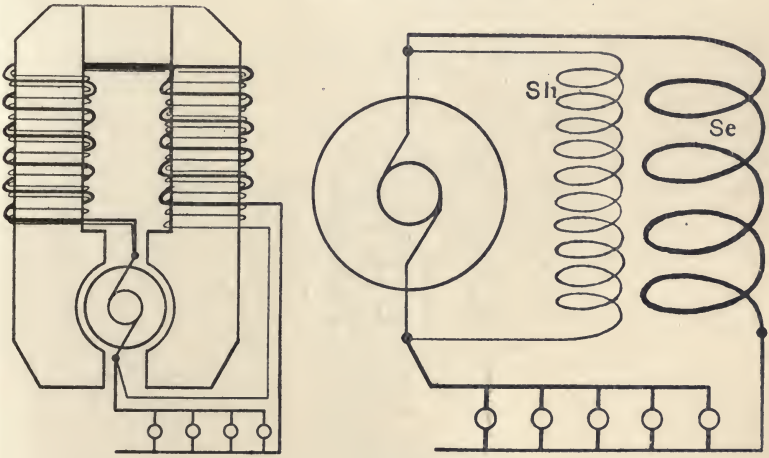


FIG. 54.

In some cases the series winding is designed to over-compensate the lost volts, and the excess allows the potential to rise slightly for full loads. When a generator gives a higher voltage at full load than on open circuit it is said to be *over-compounded*.

The following relationships, obtained by applying Ohm's Law, are important:

(*a*) *Short-shunt* compound winding (Fig. 54).

$$\left.\begin{aligned} C_a &= C + C_{sh} = C_{se} + C_{sh} \\ &= C\left(1 + \frac{r_{se} + R}{r_{sh}}\right) \quad \because \frac{C_{sh}}{C} = \frac{r_{se} + R}{r_{sh}} \\ &= C\left(\frac{r_{sh} + r_{se} + R}{r_{sh}}\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} C_{sh} &= \frac{e^1}{r_{sh}} = \frac{e + C_{se}\, r_{se}}{r_{sh}} \\ \text{where} \quad E_b = e^1 &= \text{P.D. at the brushes} \\ E_o = e &= \text{P.D.} \quad ,, \quad \text{terminals} \end{aligned}\right\}$$

$$\left.\begin{aligned} E_a &= v + e^1 = C_a\, r_a + C\,(r_{se} + R) \\ &= e + C_a\, r_a + C\, r_{se} \\ &= e\left\{1 + \frac{(r_{sh} + r_{se} + R)\, r_a}{R\, r_{sh}} + \frac{r_{se}}{R}\right\} \\ &= e\left\{1 + \frac{r_a + r_{se}}{R} + \frac{r_a\,(r_{se} + R)}{R\, r_{sh}}\right\} \end{aligned}\right\}$$

$I^2 r$

$$\left.\begin{aligned} \eta_e &= \frac{C^2\, R}{C^2\, R + C_a^{\,2}\, r_a + C^2\, r_{se} + C_{sh}^{\,2}\, r_{sh}} \\ &= \frac{1}{1 + \dfrac{r_a + r_{se}}{R} + \dfrac{R + 2\, r_a}{r_{sh}} + \dfrac{R\, r_a}{r_{sh}^{\,2}}} \end{aligned}\right\}$$

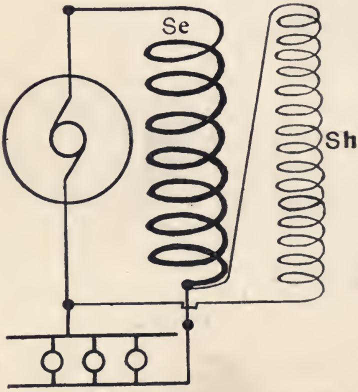


FIG. 55.

(b) *Long-shunt* compound winding (Fig. 55).

$$\left.\begin{aligned} C_a &= C_{se} = C + C_{sh} \\ &= C + \frac{e}{r_{sh}} = C \times \frac{R + r_{sh}}{r_{sh}} \end{aligned}\right\}$$

$$\left.\begin{aligned} C_{se} &= (e^1 - e)\, r_{se} \\ C_{sh} &= \frac{e}{r_{sh}} = C \times \frac{R}{r_{sh}} \end{aligned}\right\}$$

$$\left.\begin{aligned}E = v + e^1 &= C_a (r_a + r_{se}) + e \\ &= e + \frac{C (R + r_{sh})}{r_{sh}} (r_a + r_{se}) \\ &= e \left\{1 + \frac{R + r_{sh}}{R r_{sh}} (r_a + r_{se})\right\}\end{aligned}\right\}$$

$$\eta_e = \frac{C^2 R}{C^2 R + C^2_{sh} r_{sh} + C^2_a (r_a + r_{se})}$$

$$= \frac{1}{1 + \frac{R + 2(r_a + r_{se})}{r_{sh}} + (r_a + r_{se}) \frac{(R^2 + r^2_{sh})}{R r^2_{sh}}}$$

Worked Example. A short-shunt compound-wound dynamo gives 140 amperes at 105 volts. Determine r_a, r_{sh}, and r_{se}, given that $6\frac{1}{4}$ per cent. of the energy generated is used internally, 50 per cent. of which is wasted in the armature and twice as much is lost in the shunt coils as in the series coils. Also determine the total E.M.F. generated.

The useful energy given out, i.e. $140 \times 105 = 14700$ watts, is $93\frac{3}{4}$ per cent. of the energy generated,

$$\therefore \quad \text{energy generated} = \frac{100}{93{\cdot}75} \times 140 \times 105 \text{ watts}$$

$$= 15680 \text{ watts,}$$

and $$\text{energy wasted} = 15680 - 14700 = 980 \text{ watts}$$

$$\therefore \quad \text{watts wasted in armature} = \tfrac{1}{2} \text{ of } 980 = 490 \text{ watts}$$

$$\text{,, \quad shunt coils} = \tfrac{2}{3} \text{ of } 490 = \frac{980}{3} \text{ watts}$$

$$\text{,, \quad series coils} = \tfrac{1}{3} \text{ of } 490 = \frac{490}{3} \text{ watts.}$$

Now energy wasted in series coil $= C^2 r_{se}$ and $C = 140$ amperes

$$\therefore \quad r_{se} \times (140)^2 = \frac{490}{3}$$

$$\therefore \quad r_{se} = \frac{490}{3 \times 140 \times 140} = 0{\cdot}008\dot{3} \text{ ohm.}$$

And drop in volts across the series coils

$$= C\, r_{se} = 140 \times 0{\cdot}008\dot{3}$$

$$= 1\tfrac{1}{6} \text{ volts.}$$

$\therefore$ P.D. at the terminals of the shunt coils $= e_1$

$$= 105 + 1\tfrac{1}{6} = 106\tfrac{1}{6} \text{ volts.}$$

Now energy wasted in shunt coils $= e_1 \times C_{sh} = \frac{980}{3}$ watts

$$\therefore \quad C_{sh} = \frac{980}{3} \div 106\tfrac{1}{6} = \frac{980}{3} \times \frac{6}{637}$$

$$= 3{\cdot}077 \text{ amperes.}$$

Also $$e_1 = C_{sh} \times r_{sh}$$

$$\therefore \quad r_{sh} = e_1 \div C_{sh} = 106\tfrac{1}{6} \div 3{\cdot}077$$

$$= 34{\cdot}5 \text{ ohms.}$$

Again, current in the armature $= C_a = C + C_{sh}$

$$= 140 + 3{\cdot}077 = 143{\cdot}077 \text{ amperes.}$$

And energy wasted in the armature

$$= (C_a)^2 r_a = 490 \text{ watts}$$

$$\therefore \quad r_a = \frac{490}{143{\cdot}077 \times 143{\cdot}077} = 0{\cdot}024 \text{ ohm.}$$

And drop in volts in the armature $= e_2$

$$= C_a \times r_a = 143{\cdot}077 \times 0{\cdot}024$$
$$= 3{\cdot}434 \text{ volts.}$$

$$\therefore \quad \text{total E.M.F. generated} = E = e + e_1 + e_2$$
$$= 105 + 1\tfrac{1}{6} + 3{\cdot}434$$
$$= 109{\cdot}6 \text{ volts.}$$

EXERCISES VIII c.

(1) For a certain compound-wound machine we have $r_a = 0{\cdot}04$ ohm, $r_{sh} = 40$ ohms, $r_{se} = 0{\cdot}25$ ohm. If $C = 120$ amperes, the P.D. (e_1) at the brushes = 105 volts, determine (*a*) the external resistance R, (*b*) the armature current, (*c*) the E.M.F. generated, and (*d*) the electrical efficiency.

(2) Determine the energy dissipated (*a*) in the armature, (*b*) in the shunt winding, (*c*) in the series winding, and also (*d*) the energy used externally, in the above exercise.

(3) A compound-wound dynamo supplies 200 amperes when the P.D. at the brushes is 105 volts; $r_a = 0{\cdot}0244$ ohm, $r_{se} = 0{\cdot}0025$ ohm and $C_{sh} = 5{\cdot}92$ amperes. Determine (*a*) the external resistance R, (*b*) the resistance of the shunt winding r_{sh}, (*c*) the armature current, (*d*) the E.M.F. generated, and (*e*) the electrical efficiency.

(4) A compound-wound dynamo generates an E.M.F. of 114 volts. When supplying 90 amperes and R = 1·2 ohms, the electrical efficiency is 90 per cent., and the watts lost in the armature, shunt, and series windings respectively are 0·5, 0·35, and 0·15 of the total internal losses. Determine (*a*) the resistance of the armature, (*b*) the resistance of the shunt winding, (*c*) armature current, and (*d*) P.D. at the brushes.

(5) Determine the energy supplied externally, and the energy wasted in the armature, shunt coils, and series windings in the machine of Exercise 4.

CHAPTER IX

THE ELECTRIC MOTOR

Section I. Principles of Electro-dynamics.

§ 64. **Elementary Principles.** As previously explained, a current of electricity exerts a deflecting influence upon a magnetic needle; in other words, a current exerts a definite mechanical force upon a magnet placed in its vicinity. Furthermore, this action is mutual, i. e. the magnet exerts mechanical force on the current, tending to move the conductor which it traverses. This may be proved by delicately suspending a coil carrying a fairly large current, and

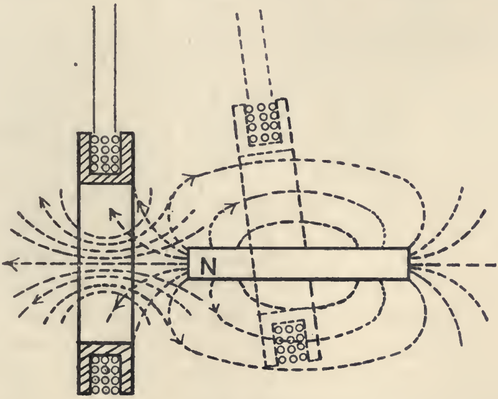


FIG. 56.

bringing a powerful bar magnet into a position along the axis of the coil: in which case the coil will be either attracted or repelled. This action may be readily explained by considering the relative dispositions of the magnetic fields set up by the current and the magnet, or, what comes to the same thing, by considering the polarity of the coil relatively to that of the magnet. With the N.-seeking pole of the magnet adjacent to the coil, and the direction of the current shown in Fig. 56, giving, according to the rule previously given for determining the polarity of a coil carrying a current, the

side of the coil adjacent to the magnet S. polarity, in which case the directions of the two sets of lines of force are coincident; consequently, according to the fundamental law of magnetism ('Like poles repel, and unlike poles attract'), the coil will be attracted towards the magnet, and, if free to do so, the coil will move so as to enclose the magnet and its field, as indicated by the dotted lines. If the current in the coil is reversed, the polarity of the coil is also reversed; the mutual action is reversed, and repulsion between the coil and magnet results.

If the magnet be placed in some position, still in the vicinity of the coil, other than along the axis of the coil, the coil then tends to rotate, and, if free to do so, will rotate and turn with a tendency to set itself with its plane at right angles to the axis of the magnet, so that attraction ultimately results. The direction of rotation consequently depends upon the relative polarities of the coil and adjacent magnet. The general rule applicable to the motion of a movable coil placed in a magnetic field was first given by Maxwell, and may be stated as follows:

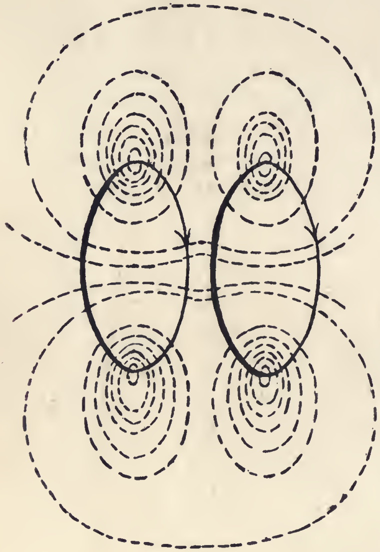


FIG. 57.

'When an electric circuit and a magnet, or two electric circuits, act mutually upon one another, they move, or tend to move, so as to set themselves that:

'(1) the direction of the magnetic lines of force coincide; and

'(2) each embraces the maximum number of lines of force possible.'

The force exerted in such cases is termed an *electro-dynamic force*, and it is upon this force that rotary motion depends in the case of electric motors. Since a current-carrying coil is an identical magnet, it is obvious that if a second coil be introduced and replace the bar magnet, and form the arrangement shown in Fig. 57, the electro-dynamic force produced tends to make the two coils approach each other, i.e. attract one another, when the directions of the currents are as indicated, in which case the directions of the two component magnetic fields are coincident; and, in consequence of the tendency for them to set themselves so as to embrace the maximum number

of lines of force possible, they tend to become co-axial. If the direction of the current in one of them be reversed repulsion occurs, and if one be free to rotate it is possible for it to twist itself round into a position so as to satisfy Maxwell's rule.

We may now apply the principles underlying the interactions taking place between a current and a magnetic field and setting up electro-dynamic force, to explain how rotary motion is produced by means of electric motors. Thus, whenever a horizontal straight conductor carrying a current is placed in a horizonal magnetic field, so that the current is in a direction at right angles to the direction of the lines of force, the resultant magnetic field above the current will be different in magnitude to that below, for obviously with the relative directions of the current and the flux, shown in Fig. 58,

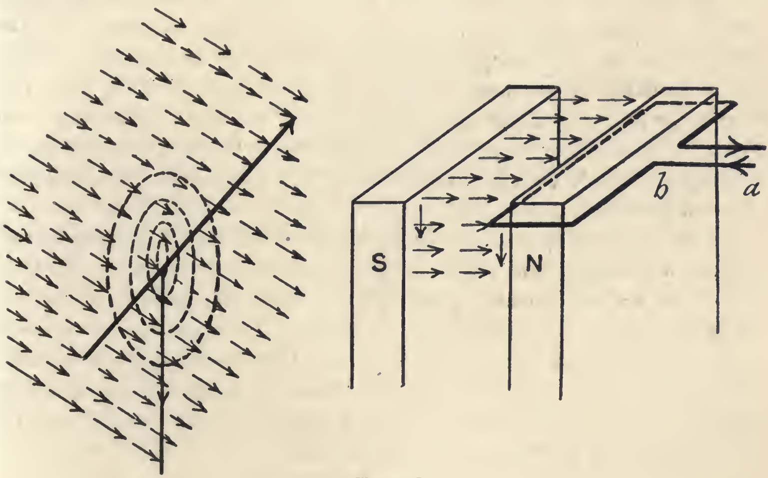


FIG. 58.

the circular and concentric lines of force due to the current coincide with and augment those of the flux above the current, and the intensity of the resultant flux is given by the *sum* of the two component intensities; whilst below the current the intensity of the resultant field is given by the *difference* of the two component intensities, one of which opposes and tends to neutralize the other. In other words, the field due to the current moves towards the flux above the current, and the conductor carrying the current experiences a force in the opposite direction, according to the law of action and reaction, and tends to move from the stronger to the weaker part of the field—i.e. that is downwards, in a direction perpendicular to itself and also to the direction of the lines of force. The three

directions—i.e. the direction of the magnetic flux, the direction of the current, and the direction of motion of the conductor—are therefore mutually perpendicular to one another, and Fleming has given a convenient hand rule for determining the direction of the mechanical force experienced by active conductors placed at right angles to the direction of lines of force, which may be stated as follows: If the forefinger of the *left hand* points in the direction of the flux, and the middle finger points in the direction of the current, then the thumb, outstretched at right angles to both the first and second fingers, will indicate the direction of motion of the conductor.

If the direction of the current or the direction of the magnetic flux be reversed, then the direction of the resulting mechanical force and motion will be reversed, and as a principle of electro-dynamics we give, as the fundamental rule governing the motion of active conductors, the following statement: Whenever a conductor carrying a continuous current is placed in a magnetic field so that the current is in a direction at right angles to the lines of force of the field, the conductor experiences a mechanical force tending to move it perpendicularly to itself, and also to the direction of the lines of force of the field.

In the absolute system of units, unit force (the dyne) is experienced when unit length (the centimetre) of unit current (ten amperes) is placed in a direction at right angles to unit magnetic (one line of force per square centimetre) field. In the C.G.S., or absolute system of units, the dyne is the unit of force, and the absolute unit of current strength is 10 amperes. The electro-dynamic force set up in consequence of the mutual action between a magnetic flux and a current of electricity is, therefore, proportional to

(1) The intensity, H, of the magnetic field.

(2) The strength of the current, C, traversing the conductor.

(3) The length of the conductor, l, at right angles to the flux.

Or, algebraically,

$$\mathrm{F} = k \times \mathsf{H} \times \mathrm{C} \times l,$$

where F = the force, and k = a dimensional constant depending on the system of units used.

If F be given in dynes, H in C.G.S. lines per square centimetre, C in absolute units of current, and l in centimetres, then $k = 1$, and

$$\mathrm{F} = \mathsf{H} \times \mathrm{C} \times l \text{ dynes}.$$

If C be given in amperes, then $k = \frac{1}{10}$, since the ampere is one-tenth of the absolute unit. And

$$\mathrm{F} = \tfrac{1}{10} \mathsf{H} \mathrm{C} l \text{ dynes}.$$

As a numerical example, let us suppose that it is required to determine the force per foot of wire tending to produce lateral displacement when a straight wire carrying a continuous current of 250 amperes lies at right angles to the magnetic flux in a field, the intensity of which is 12000 lines per square centimetre. In this case

$$l = 1 \text{ ft.} = 12 \times 2{\cdot}54 \text{ cms., since } 1 \text{ in.} = 2{\cdot}54 \text{ cms.}$$
$$= 30{\cdot}48 \text{ cms.}$$
$$\therefore \quad F = \tfrac{1}{10} \times 12000 \times 250 \times 30{\cdot}48 \text{ dynes.}$$
$$= 9144000 \text{ dynes.}$$

If the force is required in pounds weight, we must remember that one pound = 453·6 grammes, and the weight of one gramme = 981 dynes.

$$\therefore \quad F = \frac{9144000}{981 \times 453{\cdot}6} \text{ pounds weight}$$
$$= 20{\cdot}55 \text{ pounds weight.}$$

In the case of a rectangular coil of wire carrying a current and placed in a magnetic field so that its plane coincides with the direction of the lines of force of the field, its longer sides, A and B, being perpendicular to the direction of the lines of force, as shown in Fig. 59, then by Maxwell's Law we know that the coil will experience a definite electro-dynamical force tending to move it into the vertical position, so that the coil may embrace as many lines of force as possible. The direction of rotation is indicated by the arrow in Fig. 59, *a*; whilst in Fig. 59, *b*, the identical case is shown by taking two wires, A and B, of the armature of a motor, and by assuming that they form two sides of a rectangular coil, it is an easy matter to trace the cause of the tendency of the armature to rotate when the conductors are supplied with current and the field magnets are excited so as to produce the polarity of the pole-pieces, indicated in the figure, by considering the electro-dynamical force experienced by each conductor, as explained above. The direction of the magnetic field is from the N. pole to the S. pole, and the direction of the current is from the front to the back in the conductor A (as indicated by the tail of the arrow inside the circle), and from the back to the front in the conductor B (as indicated by the point of the arrow). Consequently the two conductors experience a force shown by the arrows, and since the forces are *equal* in magnitude, *parallel*, but *opposite* in direction, they constitute a *couple*, which gives rise to a twist or torque, tending to rotate the armature about the shaft.

As we have already indicated, the magnitude of the electro-dynamical force set up when a current is placed so as to be at right angles to lines of force, depends upon the intensity of the magnetic flux in which it is placed, and in practice the intensity of the

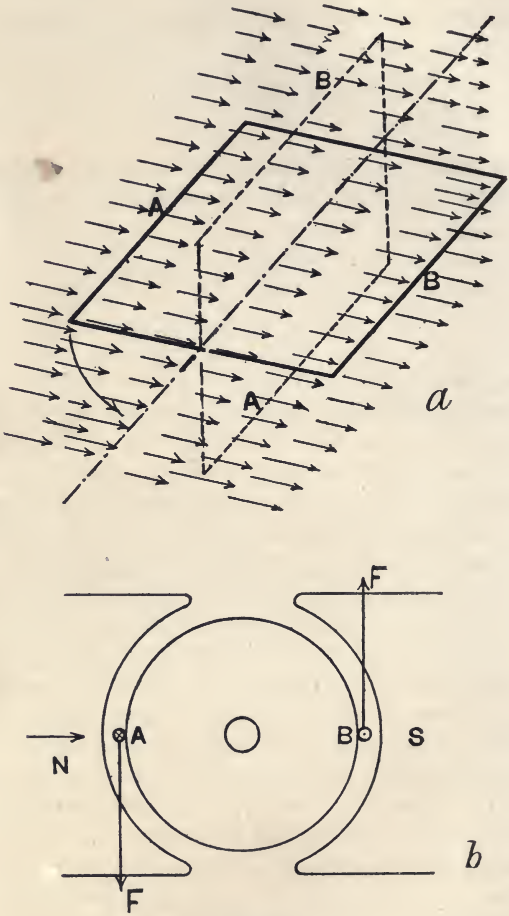


FIG. 59.

magnetic field of motors is great; and the magnetic field produced by the current traversing the armature is very much augmented by winding the conductors upon an iron core, so that the torque, or the tendency to produce rotation which results, is considerable. In

a continuous-current motor, the core of the armature is wound with a large number of conducting loops disposed uniformly round the core. The core is supported and rigidly attached to a shaft, so that it is free to rotate when the torque produced is sufficient to set up motion in consequence of the action of the electro-dynamical forces, which results when the conductors of the armature are supplied with current. Each loop, of course, tends to move, so that it may embrace the maximum number of lines of force possible, and this occurs when the plane of the coil or loop is at right angles to the direction of the lines of the field; and when a loop or coil reaches this position the electro-dynamical force is a minimum and zero, and the tendency of the coil to move forward ceases. Should the coil be carried beyond this point, an electro-dynamic force would be set up in the opposite direction, and tend to take it back into the position in which it would embrace the maximum flux. Such a position is equivalent to a dead point, so long as the direction of the current is not changed, and the expedient devised to secure a continuous rotary motion of the armature of a motor in practice is simple and effective, and is as follows: the current is led into the armature by means of brushes, and by employing a commutator the direction of the current in each coil is reversed just at the instant when the mechanical force experienced by the coil ceases, and it is obvious that the reversal of the current gives rise to a torque urging the armature round in the same direction as before.

§ 65. **Fundamental Relations.** When dealing with the principles of electro-dynamics, it was shown that a circuit traversed by a current and placed in a magnetic field, so that the direction of the current is perpendicular to the direction of the flux, tends to move so as to embrace the maximum number of lines of force possible. If the circuit moves, the electro-magnetic forces do work on the circuit, and energy must be supplied. If the circuit be fixed, the current will be constant, and its value be given by $C_1 = \frac{E}{R}$, where E is the E.M.F. of supply and R the resistance of the circuit, and the whole of the energy supplied will be converted into thermal energy. When, however, the circuit moves, in consequence of the electro-dynamic action, experiment shows that the current now traversing the circuit is not given by the ratio of the E.M.F. of supply to the resistance, but is less than C_1, the current traversing the circuit when stationary, the conditions of supply remaining the same; consequently less work is done by the source in proportion to the diminution of current. But the heat generated by the current falls off more rapidly than the work done diminishes, since the heat

produced is, by Joule's Law, proportional to the square of the current; therefore, the energy supplied by the source exceeds that corresponding to the heat generated, and the difference between them is the energy expended in doing mechanical work on the circuit.

To determine the quantitative relationships between these quantities of energy, let us suppose that AB and CD are two parallel metal conductors or rails (of negligible resistance) placed at right angles to the direction of a magnetic field, as shown in Fig. 60, and that the ends A and D are connected to a source of constant E.M.F. Then, if a metal slider, EF, be placed across and perpendicular to the rails, and also to the lines of force of the field, as shown, a complete and closed electric circuit is formed, and the slider is urged in a direction at right angles to its own length, and

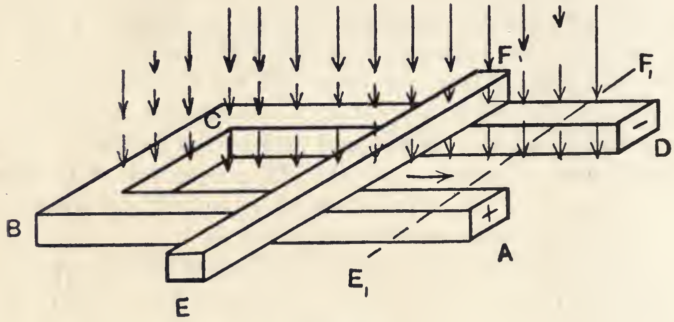


Fig. 60.

to the direction of the magnetic flux, as indicated by the arrow. In the absolute system of units, unit force (the dyne) is experienced by each centimetre of the slider if the current is of unit strength (10 amperes) and the magnetic field be of unit strength (one line of force per square centimetre); therefore, if the field be H units strength, the current C units, and the distance between the rails be l centimetres, the force experienced by the slider is

$$F = \mathrm{H}\,\mathrm{C}l \text{ dynes.}$$

If the slider moves with a velocity of v centimetres per second

$$\text{the work done per second} = \mathrm{H}\,\mathrm{C}\,lv \text{ ergs.}$$

Let us now suppose that the slider is at $E_1 F_1$, t seconds after it was at EF, and that the total flux enclosed by the circuit AEFD was N_2, whilst that enclosed by the circuit AE_1F_1D is N_1, then

$N_1 - N_2 = H\,lv \times t$, and this number of lines is cut by the slider by its movement, and

$$\text{work done in } t \text{ seconds} = C \times H\,lvt \text{ ergs}$$
$$= C \times (N_1 - N_2) \text{ ergs}$$

or, in words, the measure of the work done in moving a current across a magnetic flux is given by the product of the current into the total number of the lines of force cut during the movement. We, therefore, have the following relationships—

$$\text{work done by electromagnetic forces in } t \text{ seconds} = C\,(N_1 - N_2) = W$$
$$\text{heat produced in } t \text{ seconds} = C^2 R\,t$$
$$\text{energy supplied by the source} = CEt = W$$

where E is the electrical pressure (assumed constant) maintained between the points C and D, and R is the resistance of the slider, then

$$CEt = C^2Rt + C\,(N_1 - N_2)$$

and
$$E = CR + \frac{N_1 - N_2}{t}$$

or
$$CR = E - \frac{N_1 - N_2}{t}$$

but CR is the electrical pressure or the potential difference utilized to send the current through the circuit, and may be termed the effective or available electrical pressure, which is obviously less than the E.M.F. of supply by an amount $\frac{N_1 - N_2}{t}$. Then again

$$C = \frac{E - \dfrac{N_1 - N_2}{t}}{R}$$

which is an equation giving in an algebraic form Ohm's Law, and indicating the existence of a new or counter E.M.F. equal to $\frac{N_1 - N_2}{t}$, which neutralizes an equal amount of the E.M.F. of supply. In other words, $\frac{N_1 - N_2}{t}$, which is clearly the time-rate of change in the flux corresponding to the movement of the slider, is the measure of an electromotive-force set up in return for the work done on the slider in varying the flux enclosed. This, in fact, was the discovery made by Faraday in 1831, which was explained in Chapter VII.

When the length of the conductor (as in an armature) is great, and the magnetic field of ordinary strength (12000 lines per sq. cm.), the tendency to produce motion, or, in other words, the drag on the armature is approximately equal to the pull of an ounce weight

per foot length of the conductor for every ampere traversing the conductor, from which it is obvious that in the case of large motors the turning moment may be considerable, and it is in consequence of this electromagnetic action which takes place between currents of electricity and a magnetic field that a motor can do work. As we have previously remarked, a dynamo is a reversible machine, and as a consequence an electric motor, when working, is a dual machine, for when the armature rotates it is at once a dynamo and a motor. On the one hand it does mechanical work, and on the other the armature becomes the seat of an electromotive-force, since an electrical pressure is always induced when an armature rotates in a magnetic field. In fact it is a characteristic feature of the action of every motor that this induced pressure is of the nature of a back pressure, and is termed the counter-electromotive-force (C.E.M.F.) of the armature. According to the Principles of the Conservation of Energy, and also to Lenz's Law, this must be so, and the counter-E.M.F. is not a detrimental factor, but an essential element in the working of a motor, and, as will be evident presently, it plays a most important part in the economical working of all motors. If we denote this back pressure by ϵ, and the pressure of supply by E, the effective pressure available for sending a current through the armature will be denoted by $E-\epsilon$; consequently the current traversing the armature will be given by

$$C_a = \frac{E-\epsilon}{r_a}$$

To simplify matters it will be convenient to assume that the field magnetism is produced by a separate source of supply, or, in other words, to assume that the motor under consideration is a separately-excited machine, and that the magnetic field is constant. A moment's consideration is sufficient to show that the strength of the armature current depends upon the magnitude of the C.E.M.F. more than upon the resistance of the armature, which is always small. It is for this reason that a starting resistance is inserted in the armature circuit when starting the motor, for since the armature is then at rest, the back pressure is zero, and the initial rush of current would be excessive. The counter-E.M.F. is also a factor of the output of a motor, as the following deduction clearly proves. The Principles of the Conservation of Energy imply that

Energy supplied = energy spent as work + energy wasted as heat

or $\qquad W = w +$ energy wasted as heat.

By expressing these quantities of energy in watts we have

$$EC_a = w + C_a^2 r_a$$

$$\therefore \quad C_a^2 r_a = EC_a - w$$

and

$$C_a r_a = E - \frac{w}{C_a}$$

$$\therefore \quad C_a = \frac{E - \dfrac{w}{C_a}}{r_a}$$

Now by Ohm's Law it is evident that $\left(E - \frac{w}{C_a}\right)$ is the effective E.M.F. sending the current through the armature, consequently $\frac{w}{C_a}$ is the counter-E.M.F. produced by the rotation of the armature. In other words

$$\frac{w}{C_a} = \epsilon$$

$$\therefore \quad \text{energy transformed into work} = w = C_a \epsilon$$

and the above expression for the armature current is as before

$$C_a = \frac{E - \epsilon}{r_a}$$

The work done is thus proportional to the counter-E.M.F., and is measured by the product of the armature current into the back pressure. From first principles it is evident that work done by a motor is determined by the factors—the armature current, the magnetic field, the number of armature conductors, and upon the number of revolutions made by the armature; and, as we have previously shown, the magnitude of an induced pressure depends solely upon the three last factors; consequently the work done must be proportional to the back pressure set up.

The back pressure is also the measure of the efficiency of a motor, since

$$\text{Energy supplied} = EC_a$$

$$\text{Energy transformed} = \epsilon C_a$$

$$\text{And efficiency} = \frac{\text{output}}{\text{input}} = \frac{\epsilon C_a}{EC_a} = \frac{\epsilon}{E}$$

This is only true so far as the electrical efficiency is concerned, and if the internal frictional losses—such as friction, hysteresis, and eddy-current losses—are negligibly small. If the pressure of supply be assumed to be constant, it follows that the output per ampere will increase as the C.E.M.F. increases; then, again, the current decreases as the back pressure increases, and this means increased efficiency.

By means of the relation $W = C_a \epsilon$, we can measure electrically the work imparted to the shaft of an electric motor, and it is now

necessary to investigate the measure of the mechanical energy given out. It is well known that the two components of the mechanical activity expended as work are (1) speed of rotation, and (2) the force exerted tangentially to the shaft or pulley; in other words, power or rate of working is measured by taking the product of the two factors—force and velocity. The speed of the rotating armature may be expressed in two ways—by stating (1) the number of revolutions, n, made per minute, and (2) the angular velocity of a point on the shaft or pulley in radians per second. The radian is that angle whose arc is equal to the radius, and if $\omega =$ angular speed, then since there are 2π radians described per revolution, and the armature makes n revolutions per minute,

$$\omega = \frac{2\pi n}{60} \text{ radians.}$$

Torque. The force exerted tangentially to the shaft is the measure of the drag on the armature conductors, which gives rise to what is known as the mechanical couple, the moment of the rotating force about the axis of the shaft, or turning moment of the rotating shaft, technically known as the *torque* of the motor. Torque is defined as that which produces or tends to produce torsion, and is measured by the tangential force exerted at unit radius. Torque is usually denoted by the letter T, which is equal to the tangential force when the radius is of unit radius, and to $f \times r$ for shafts or pulleys of radius r, or

$$\mathrm{T} = f \times r.$$

When f is given in dynes and r in centimetres, the torque is expressed in dyne-centimetres, whilst if f is given in pounds-weight and r in feet, the torque T is expressed in pound-feet.

Now since the mechanical activity is measured by the product of the tangential force f and the circumferential velocity v,

$$\mathrm{W} = f \times v = f \times r \times \frac{v}{r}$$

but $\frac{v}{r}$ is by definition the number of radians.

$$\therefore \quad \mathrm{W} = \mathrm{T} \times \omega = \text{work done per second.}$$

But

$$\omega = \frac{2\pi n}{60}$$

$$\therefore \quad \mathrm{W} = \frac{2\pi n \mathrm{T}}{60}$$

Connecting the electrical and mechanical measures of the activity, we have

$$C_a \epsilon = T\omega = \frac{2\pi n T}{60}$$

But $C_a \epsilon$ is given in watts, therefore $\frac{2\pi n T}{60}$ will be expressed in watts. 1 H.P. = 746 watts = 550 foot-pounds per second. Therefore $\frac{746}{550}$ watts = 1 foot-pound. Therefore to change to foot-pounds and to express the torque in pound-feet, we have

$$\frac{2\pi n T}{60} = C_a \epsilon \times \frac{550}{746} \text{ foot-pounds}$$

and

$$T \text{ pound-feet} = \frac{C_a \epsilon}{2\pi n} \times \frac{550}{746} \times 60$$

But by the laws of induction

$$\epsilon = \frac{N \mathfrak{C} n}{10^8 \times 60} \text{ volts.}$$

$$\therefore \quad T = \frac{C_a N \mathfrak{C} n}{2\pi n 10^8} \times \frac{550}{746}$$

$$= C_a N \mathfrak{C} \times \frac{550}{2\pi \times 746 \times 10^8}$$

$$= \frac{C_a N \mathfrak{C}}{8{\cdot}5257 \times 10^8}$$

$$= 0{\cdot}11733\, C_a N \mathfrak{C} \times 10^{-8} \text{ pound-feet.}$$

The measure of the torque, T, of an electric motor expressed as a moment of a force, f pounds-weight, acting at a distance of r feet from the axis of rotation, as deduced above, is a most important factor upon which the choice of a type of motor for a particular kind of work depends, inasmuch as torque and speed are the controlling factors of all kinds of machine work. It will be profitable, therefore, to consider briefly how a desired torque and speed may be obtained, and to do this it will be necessary to trace the connexion between the torque, speed, and various elements in the design of a motor. In the formula, $T = 0{\cdot}11733\, C_a N \mathfrak{C} 10^{-8}$ pound-feet. The quantities $\mathfrak{C}$, N and 10^{-8} are constants, $\mathfrak{C}$ being the number of conductors on the armature, and N the total magnetic flux passing through the armature. They are obviously quantities depending upon the construction of the machine, and since the E.M.F. induced in a rotating armature of a dynamo is given by

$$E = \frac{N \mathfrak{C} n}{10^8 \times 60} \text{ volts}$$

$$N \mathfrak{C} 10^{-8} = \frac{E}{n/60} = \text{volts induced per revolution.}$$

for which reason $\mathsf{N}\mathcal{Z}10^{-8}$ has been termed by Professor Carus-Wilson the 'induction-factor' of the machine, and is usually denoted by the letter M. The magnitude of the induction-factor does not depend upon the speed, and is readily found by running the machine on open circuit when separately excited, so as to obtain the magnetic flux N, and by measuring the pressure induced with an electrostatic voltmeter; then, if E be the E.M.F. induced when the armature makes n revolutions per second

$$\mathsf{M} = \frac{\mathrm{E}}{n} \text{ or } \mathrm{E} = \mathsf{M}n.$$

By substituting M for $\mathsf{N}\mathcal{Z}10^{-8}$ in the fundamental equation for torque, we have

$$\begin{aligned} \mathrm{T} &= 0{\cdot}11733\,\mathrm{C}_a\,\mathsf{N}\mathcal{Z}10^{-8} \text{ pound-feet} \\ &= 0{\cdot}11733\,\mathrm{C}_a\,\mathsf{M} \text{ pound-feet,} \end{aligned}$$

which is a very convenient expression for finding the numerical measure of the torque, since M may be determined separately for a given excitation and speed. It must, however, be remembered that these expressions give the torque as the turning moment corresponding to a certain amount of electrical energy converted into mechanical energy; but in no case can this torque be available at the pulley of a motor, since energy is lost internally. Suppose we require to determine the torque which corresponds to the energy converted in the case of a motor running at 600 revolutions per minute, when 250 amperes pass through the armature, a previous test having shown that 140 volts are induced at this speed when driven as a dynamo on open circuit. Then

$$\mathsf{M} = \frac{\mathrm{E}}{n/60} = \frac{140}{600/60} = 14.$$

and

$$\begin{aligned} \text{Torque} = \mathrm{T} &= 0{\cdot}11733\,\mathrm{C}_a\,\mathsf{M} \\ &= 0{\cdot}11733 \times 250 \times 14 \\ &= 410{\cdot}655 \text{ pound-feet.} \end{aligned}$$

If the torque be known in pound-feet, then the induction-factor M is found as follows:

$$\mathsf{M} = \frac{\mathrm{T}}{{\cdot}11733\,\mathrm{C}_a}$$

When a motor is working on load, the forces acting on the shaft may be divided into positive and negative forces. The armature drag, for instance, is positive since it assists in producing motion, whilst the load and various frictions introduce negative forces which oppose the motion. Thus the energy supplied to a motor is transformed into other forms of energy, which may be specified separately as follows:

(1) Useful mechanical energy, which may be expressed as so many B.H.P. available at the pulley, or as a certain torque exerted at the pulley overcoming the opposing torque corresponding to the load.

(2) The C^2R losses, corresponding to the heat produced in the wires.

(3) Friction at the bearings and brushes, and also wind resistance, the sum of which is equivalent to a certain resisting torque or negative force resisting motion.

(4) Eddy current losses in the armature and pole-pieces which resist the motion of the armature, in consequence of the inductive effects set up.

(5) Hysteresis losses, which result in consequence of a lag taking place during magnetization and demagnetization of the iron in the armature and pole-pieces; consequently it is found that a definite torque is required to rotate a mass of iron in a magnetic field, the magnitude of which depends upon the intensity of the field, the volume and quality of the iron rotated, and if the excitation be constant the resisting torque—termed hysteresis torque—will be constant. The losses in (3) and (4), however, increase with increase of speed. In what follows the term 'load' will denote those forces which resist the motion, and in many cases it will be convenient to express it as so many pound-feet. When the positive and negative torques exactly balance one another, motion, if it occurs, is uniform; if the positive torque exceed the opposing or negative torque the excess is available for producing acceleration.

It is quite an easy matter to obtain the torque actually concerned in doing useful work if the B.H.P. be determined by means of some kind of dynamometer; thus, let B.H.P. denote the brake-horsepower of a motor running at n revolutions per minute, then the torque at the shaft, T_s, is

$$T_s = \frac{33000}{2\pi n} \text{ B.H.P.}$$

$$= \frac{5250}{n} \text{ pound-feet per brake-horsepower.}$$

This is the moment of the forces tending to turn the shaft, and is equal to a force of $\frac{5250}{n}$ pounds acting at a radius of 1 ft. per B.H.P. If the pulley is r feet radius, the tangential force at the pulley is $\frac{5250}{n} \times \frac{1}{r}$ pounds per B.H.P. If we denote the torque corresponding to the electrical energy transformed by T_a, then

$T_a - T_s$ = torque corresponding to the losses (3), (4), and (5) above.

The question of speed is also an important factor in motor

working, and is one which has to be considered in detail with each kind of motor. We may, however, point out that in a motor with constant magnetic field, the E.M.F. at the armature terminals is split up into two components, one ϵ, which is equal to and neutralizes the counter-E.M.F. set up, the other e_a is equal to $C_a r_a$, and is the pressure utilized in sending the current through the armature. From first principles, therefore

$$E_a = \epsilon + e_a = \epsilon + C_a r_a$$

and energy supplied $= E_a C_a = C_a(\epsilon + C_a r_a)$

but $C_a \epsilon = T_a \omega = 0{\cdot}11733\, C_a M\, \omega$

and $M n = \epsilon$,

or, in words, the C.E.M.F. is proportional to the speed, and it follows that the armature will run at such a speed that the

Input = output (energy used + energy wasted),

and for a certain input there is a certain speed.

To indicate the direction of rotation, we shall find it necessary to compare it with that which occurs when the machine is driven as a generator, the terminals retaining the same positive and negative polarity in both cases. By applying Fleming's hand rules, previously given, it will be possible to determine the relative direction of current and of the motion of the armature for the generator and motor. In Fig. 58 a conductor is placed so as to be at right angles to the direction of a magnetic field, and if the conductor be moved in a vertical direction, the action is that of a generator. Upon moving the conductor vertically upwards, the direction of the induced current is from the back to the front—i.e. from b to a. If, however, a current is sent along the conductor from the back to the front, the coil will be urged vertically downwards. We therefore have the general rule, 'If a dynamo be run as a motor and be supplied with current so that the direction of the magnetic field remains the same, and the current traverses the armature in the same direction as when running as a dynamo, then the direction of rotation as a motor will be opposite to what it was when running as a dynamo.'

Since $T = 0{\cdot}11733\, C_a M$, it follows, if the magnetic flux is constant, that the induction factor M will be constant, consequently

T is proportional to C_a, the armature current.

But $$C_a = \frac{E_a - \epsilon}{r_a}$$

therefore $$T = {\cdot}11733\, M \frac{(E_a - \epsilon)}{r_a}$$

$$= {\cdot}11733 \frac{M}{r_a}(E_a - nM)$$

and T is zero if $n\mathsf{M}$ or $\epsilon = E_a$. This condition corresponds to the case of the motor running empty when the speed n is a *maximum*, and such that ϵ approaches E_a. On the other hand, the torque T is a maximum when n is zero, and then

$$T = \cdot 11733\,\mathsf{M}\frac{E_a}{r_a}$$
$$= \cdot 11733\,\mathsf{M}\,C_a{}^1 \text{ (by Ohm's Law),}$$

$C_a{}^1$ being the maximum value of the armature current which occurs when the armature is at rest, and $n = 0$. Then again

$$T\,r_a = \cdot 11733\,E_a\mathsf{M} - \cdot 11733\,n\mathsf{M}^2$$

and
$$n = \frac{\cdot 11733\,E_a\,\mathsf{M} - T\,r_a}{\cdot 11733\,\mathsf{M}^2}$$
$$= \frac{E_a}{\mathsf{M}} - \frac{T\,r_a}{\cdot 11733\,\mathsf{M}^2} \quad \ldots\ldots\ldots \quad (a)$$

This formula gives the connexion between the speed n, E.M.F. of supply E_a, the induction factor M, the torque T, and the resistance of the armature r_a. It is also instructive to trace the connexion between the speed and the armature current C_a.

Since
$$E_a = \epsilon + C_a r_a \text{ and } \epsilon = \mathsf{M}\,n$$
$$\mathsf{M}\,n = E_a - C_a r_a$$
$$\therefore \quad n = \frac{E_a - C_a r_a}{\mathsf{M}} \quad \ldots\ldots\ldots\ldots \quad (\beta)$$

Consequently if E_a, r_a, and M are fixed quantities, the dependence of the speed upon the armature current is obvious. Equation (a) shows, as one would expect, that the speed increases as the torque diminishes; but since the torque is proportional to the armature current, the speed decreases as the armature current increases. This is in agreement with what we have shown previously, for

$$C_a = \frac{E_a - \epsilon}{r_a}$$

and ϵ is proportional to the speed.

The following deductions from the relations given in equations a and β are also obvious.

(1) n is positive if E_a is greater than $C_a r_a$.

(2) When the motor is running empty, i.e. when the load may be taken as zero, C_a is a minimum, and $C_a r_a$ may be taken as zero; then $n = \frac{E_a}{\mathsf{M}}$ = the maximum value of the speed of the motor.

As shown in the equations a and β, the induction-factor plays a most important part in the value of the torque and speed, and as it

is a factor depending upon the construction of the motor, it will be useful to indicate how the value of the induction factor M is connected with the speed and torque or output of the motor. To simplify matters, let us assume that the pressure of supply E_a, the resistance of the armature r_a, the revolutions per second n, and the output W in watts be given, then

since $$n = \frac{E_a}{M} - \frac{T r_a}{\cdot 11733\, M^2} = \frac{E_a}{M} - 8{\cdot}5229 \frac{T r_a}{M^2}$$

$$M^2 - \frac{E_a}{n} M + \frac{8{\cdot}5229\, T r_a}{n} = 0$$

and $$M = \frac{E_a \pm \sqrt{E_a^2 - 4 \times 8{\cdot}5229\, T r_a n}}{2n}$$

$$= \frac{E_a}{2n}\left\{1 \pm \sqrt{1 - \frac{4 \times 8{\cdot}5229\, T r_a n}{E_a}}\right\}.$$

Then again $M = \dfrac{E_a - C_a r_a}{n}$ and $W = C_a \epsilon = C_a M n$

$$\therefore \quad M = \frac{E_a}{n} - \frac{W r_a}{M n^2} \text{ or } M^2 - \frac{E_a}{n} M + \frac{W r_a}{n^2} = 0$$

and $$M = \tfrac{1}{2}\frac{E_a}{n}\left\{1 \pm \sqrt{1 - \frac{4\, r_a W}{E_a^2}}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (\gamma)$$

if W = output in watts

and $$M = \frac{E_a}{2n}\left\{1 \pm \sqrt{1 - 2984\, \frac{r_a\, \text{H.P.}}{E_a^2}}\right\} \quad . \; . \; . \; . \quad (\delta)$$

If H.P. = output in H.P. (horse-power)

Note 2984 = 4 × 746.

Equations (γ) and (δ) have two values of M, and as the higher value of M corresponds to the smaller current, the positive sign must be taken in the values of M given by γ and δ. For the present the power lost internally as friction is included in the values of T, W, and H.P.

Worked Examples. (1) A series motor is connected between the mains of a 100-volt circuit, and runs at such a speed that the back pressure developed is 75 volts. The resistance is 1·25 ohms; determine (*a*) the current, (*b*) the power converted into mechanical energy (neglecting frictional losses), (*c*) energy wasted, and (*d*) the efficiency.

$$C_a = \frac{E - \epsilon}{R} = \frac{100 - 75}{1{\cdot}25}$$

$$= \frac{25}{1{\cdot}25} = 20 \text{ amperes.}$$

Energy converted into mechanical energy is $w = C\epsilon$

$$\therefore \quad w = 20 \times 75 = 1500 \text{ watts.}$$

Since $E = \epsilon + CR$ (for a series motor)

$$\text{Energy supplied is } EC = C\epsilon + C^2 R.$$

And energy wasted is $EC - C\epsilon = C^2 R$

$$= (20)^2 \times 1{\cdot}25$$
$$= 500 \text{ watts.}$$

$$\text{Electrical efficiency} = \frac{C\epsilon}{CE} = \frac{CE - C^2R}{CE}$$
$$= \frac{\epsilon}{E} = \frac{75}{100} = 75 \text{ per cent.}$$

(2) Determine the electrical efficiency of a series motor which develops 1 H.P. of mechanical energy when the current is 12 amperes and the total resistance $2\frac{1}{2}$ ohms.

Mechanical energy developed $= C\epsilon = 1$ H.P. $= 746$ watts.

Electrical energy supplied $= CE = C(\epsilon + CR)$

$$\therefore \quad \text{Electrical efficiency} = \frac{\epsilon}{\epsilon + CR}.$$

But

$$\epsilon = \frac{746}{C} = \frac{746}{12} = 62{\cdot}2 \text{ volts}$$

$$\therefore \quad \text{Electrical efficiency} = \frac{62{\cdot}2}{62{\cdot}2 + 12 \times 2\ 5}$$
$$= \frac{62{\cdot}2}{92{\cdot}2} = 67{\cdot}4 \text{ per cent.}$$

(3) A series motor is worked from 220-volt mains, and the total resistance is 2·75 ohms. Determine the available power in watts when the current is 30, 40, 50, and 60 amperes respectively.

$$\text{Energy supplied} = EC.$$
$$\text{Energy wasted} = C^2R.$$
$$\therefore \quad \text{Energy available} = EC - C^2R.$$

(*a*) When $C = 30$ amperes

$$\text{power available} = 220 \times 30 - (30)^2 \times 2{\cdot}75$$
$$= 6600 - 2475$$
$$= 4125 \text{ watts.}$$

(*b*) When $C = 40$ amperes

$$\text{power available} = 220 \times 40 - (40)^2 \times 2{\cdot}75 = 4400 \text{ watts.}$$

(*c*) When $C = 50$ amperes

$$\text{power available} = 220 \times 50 - (50)^2 \times 2{\cdot}75 = 4125 \text{ watts.}$$

(*d*) When $C = 60$ amperes

$$\text{power available} = 220 \times 60 - (60)^2 \times 2{\cdot}75 = 3300 \text{ watts.}$$

(4) Determine the power transmitted in terms of the tensions of a belt.

In Fig. 61 *a*, M is the pulley of a motor which drives a pulley, C, on a counter-shaft by means of a belt. Now, if we consider the arrangement shown in 61 *b*, in which two unequal weights, t and s, suspended from the pulley, C, by a cord, produce rotation in the direction indicated, it will be obvious that two forces, T_t and T_s, are acting on the pulley, C, but that the torque which gives rise to the rotation is produced by the difference of the tensions T_t and T_s or $T_t - T_s$ (T_t being greater than T_s). Therefore, the power given to the pulley is the difference of the work done in unit time *by* the greater weight falling, and that done *on* the smaller weight in raising it. In Fig.

61 *a* these two tensions are marked, T_t being the tension of the tight or driving side, and T_s the tension of the slack or following side. If the velocity of a point on the belt be v feet per second, then the power transmitted by a belt is as follows :—

$$\text{Power transmitted} = \frac{(T_t - T_s)v}{550} \text{ H.P.}$$

$$= \frac{2\pi n r\,(T_t - T_s)}{550} \text{ H.P.}$$

(5) The tangential force exerted at the pulley of a motor is 226·8 kilogrammes weight, and the diameter of the pulley is 0·9144 metre. Determine the torque in dyne-centimetres.

Since $T = f \times r$

$$\begin{aligned} \text{torque} &= 226{\cdot}8 \times \tfrac{1}{2} \text{ of } 0{\cdot}9144 \text{ kilogramme-metres} \\ &= 226{\cdot}8 \times 1000 \times 0{\cdot}4572 \times 100 \text{ gramme-centimetres} \\ &= 226800 \times 45{\cdot}72 \times 981 \text{ dyne-centimetres} \\ &= 10172279376 \text{ dyne-centimetres.} \end{aligned}$$

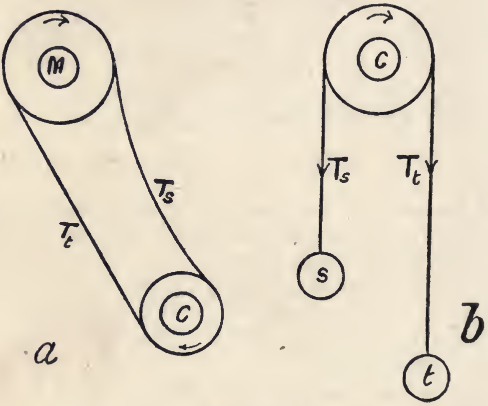


FIG. 61.

(6) A certain motor with drum armature 12 inches in diameter with 120 conductors makes 980 revolutions per minute. The resistance of the armature is 0·025 ohm, that of the shunt coil being 30 ohms. Given that N = 6750000 lines, and that the current supplied is $203\frac{2}{3}$ amperes at 110 volts, determine the average force on each active armature conductor, the horse-power given out, and the efficiency, neglecting frictional losses. The number of active conductors may be assumed to be 90.

(*a*) *Determination of average force per active conductor.*

$$\text{The torque} = {\cdot}11733\, C_a N \mho \div 10^8$$

and $C = C_a + C_{sh}$, $\therefore$ $C_a = C - C_s$

$$= 203\tfrac{2}{3} - \frac{110}{30}$$

$$= 200 \text{ amperes.}$$

Note. $$C_{sh} = \frac{\text{E.M.F. supplied}}{r_{sh}}$$

$$\therefore \quad T = \frac{\cdot 11733 \times 200 \times 6750000 \times 120}{10^8}$$

$$= 189 \cdot 54 \text{ pound-feet,}$$

but T = total peripheral force $\times$ radius,

$$\therefore \quad \text{total peripheral force} = \frac{189 \cdot 54}{0 \cdot 5} = 379 \cdot 08 \text{ pounds,}$$

and $$\text{average force per active conductor} = \frac{379 \cdot 08}{90} \text{ pounds}$$

$$= 4 \cdot 2 \text{ pounds.}$$

(*b*) *Determination of horse-power given out.*

$$\text{The volts lost in the armature} = C_a \times r_a$$
$$= 200 \times 0 \cdot 025$$
$$= 5 \text{ volts,}$$

and $$\text{C.E.M.F.} = 110 - 5 = 105 \text{ volts} = \epsilon$$

and $$W = C_a \times \epsilon = 200 \times 105 = 21000 \text{ watts}$$

and $$\text{output} = \frac{21000}{746} = 28 \text{ H.P.}$$

(*c*) *Determination of efficiency.*

$$\eta = \frac{C_a \, \epsilon}{C \, E} = \frac{200 \times 105}{203\frac{2}{3} \times 110}$$
$$= 93 \cdot 8 \text{ per cent.}$$

(7) Determine the magnetic flux passing through the armature of a 4-pole motor having 368 conductors. The output is 250 H.P. at 500 revolutions per minute when 275 amperes pass through the armature. What is the turning moment of the shaft?

(*a*) *Determination of magnetic flux.*

$$\text{The C.E.M.F.} = \epsilon = \frac{250 \times 746}{275} \text{ volts}$$

$$= \frac{2\,N \, . \, \mathfrak{T} \, . \, n}{60 \times 10^8} \, .$$

The constant 2 is introduced because N is the flux between each pair of poles.

$$\therefore \quad \frac{250 \times 746}{275} = \frac{2\,N \times 368 \times 500}{60 \times 10^8}$$

and $$2\,N = \frac{250 \times 746 \times 60 \times 10^8}{275 \times 368 \times 500}$$

$$= 2211400 \text{ lines.}$$

(*b*) *Determination of torque.*

$$T = \frac{\cdot 11733 \times C_a \times 2\,N \times \mathfrak{T}}{10^8}$$

$$= \cdot 11733 \times 275 \times \frac{250 \times 746 \times 60 \times 10^8}{275 \times 368 \times 500} \times 368 \div 10^8$$

$$= 2626 \text{ pound-feet.}$$

Note. The torque may also be found as follows :—

$$\text{Work per revolution} = \frac{\text{H.P.} \times 33000}{n} \text{ foot-pounds.}$$

$$= 2\,\pi\,T \text{ pound-feet.}$$

$$\therefore \quad T = \frac{\text{H.P.} \times 33000}{2\,\pi \times 500} = \frac{250 \times 33000}{2 \times 3{\cdot}1416 \times 500}$$
$$= 2626 \text{ pound-feet.}$$

(8) A certain motor has an induction factor $M = 10$, the resistance of the armature being 0·75 ohm. If the field magnet coils be separately excited, and M kept constant, then if the E.M.F., E_a, at the terminals of the armature be 120 volts, determine the speed and C.E.M.F. for a load corresponding to a torque of 100 pound-feet.

The maximum armature current $C_a{}^1 = \frac{120}{0{\cdot}75} = 160$ amperes, therefore the maximum torque when starting from rest is

$$T = 0{\cdot}11733 \text{ M } C_a{}^1 = 0{\cdot}11733 \times 10 \times 160 = 187{\cdot}728 \text{ pound-feet.}$$

When the load put upon the machine corresponds to a torque of 100 pound-feet, the armature current C_a is

$$C_a = \frac{T}{0{\cdot}11733 \text{ M}} = \frac{100}{0{\cdot}11733 \times 10}$$
$$= 85{\cdot}229 \text{ amperes.}$$

and the speed is $$n = \frac{E_a - C_a r_a}{M} \times 60 \text{ r.p.m.}$$
$$= \frac{120 - 85{\cdot}229 \times {\cdot}75}{10} \times 60 = 336{\cdot}5 \text{ r.p.m.}$$

The drop in volts in the armature

$$= C_a r_a = 85{\cdot}229 \times {\cdot}75 = 63{\cdot}922 \text{ volts.}$$

The C.E.M.F. $\epsilon = 120 - 63{\cdot}922 = 56{\cdot}078$ volts,

Note. $$\epsilon = M n = 10 \times \frac{336{\cdot}5}{60}$$
$$= 56{\cdot}08 \text{ volts.}$$

If the load be thrown off, then the speed increases and tends to approach the maximum speed possible, i. e.

$$n_1 = \frac{\epsilon}{M} = \frac{E_a}{M} = \frac{120}{10} = 12 \text{ revolutions per second}$$
$$= 12 \times 60 = 720 \text{ r.p.m.,}$$

which is more than twice the speed corresponding to the torque of 100 pound-feet.

(9) A motor is required to exert a torque of 75 pound-feet, and run at a speed of 450 revolutions per minute. Assuming 0·5 ohm as the armature-resistance, and the pressure of supply 400 volts, determine the value of the induction factor.

$$M = \frac{E_a + \sqrt{E_a{}^2 - 4 \times 8{\cdot}5229 \text{ T } r_a n}}{2n}$$

$$= \frac{400 + \sqrt{(400)^2 - 34{\cdot}0916 \times 75 \times 0{\cdot}5 \times 7{\cdot}5}}{2 \times \frac{450}{60}}$$

$$= 52{\cdot}52.$$

EXERCISES IX A.

(1) The efficiency of an electric motor supplied at a pressure of 100 volts is 95 per cent. The resistance is 0·01 ohm; determine the output in horse-power.

(2) A motor is driven from 100-volt mains, and gives an actual output of 4 H.P. If this is 90 per cent. of the power supplied, find the number of amperes required.

(3) Determine the electrical efficiency of a series motor supplied by current from 100-volt mains, if the resistance of the armature is 0·15 ohm and that of the field-magnet coils is 0·25 ohm, and 1·6 H.P. is given out.

(4) A motor of ·45 ohm resistance is connected to a dynamo by leads having a resistance of 1·05 ohms. The terminal voltage of the generator is 105 volts when the current is 4 amperes; determine the counter-E.M.F. and the D.P. maintained between its terminals.

(5) The horse-power available at the shaft of a series motor is 6 H.P., when a current of 50 amperes at 120 volts is supplied. Determine the electrical efficiency, (*a*) neglecting frictional losses, (*b*) assuming the frictional losses to be equivalent to 300 watts.

(6) What is the total resistance of the above machine (assume frictional losses as 300 watts), and how much energy is wasted in working the motor?

(7) A series motor is run from 100-volt mains, and gives an actual output of 4 H.P. If this is 90 per cent. of the power supplied, determine the number of amperes required, (*a*) neglecting frictional losses, (*b*) assuming the frictional losses to be equivalent to 270 watts.

(8) The commercial efficiency of a series motor run from 100-volt mains is 85·5 per cent. The resistance of the armature and field-magnet coils is 0·5 ohm. Determine (*a*) the current, (*b*) the power available at the pulley if the frictional losses are equivalent to 500 watts.

(9) A series motor, the internal resistance of which is 0·05 ohm, receives 40 amperes from a 100-volt circuit, and the commercial efficiency is 85·5 per cent. when the power available at the pulley is 3420 watts. Determine the back pressure and the frictional losses.

(10) Determine the electrical efficiency of a series motor supplied with current from 100-volt mains if the resistance of the armature is 0·15 ohm, and that of the field-magnet coil 0·25 ohm, when 1·25 H.P. is available at the pulley and the frictional losses are equivalent to 67·5 watts. Also determine the current required.

(11) A motor gives a brake horse-power of 7 H.P. when the frictional losses are equivalent to 300 watts, and runs at such a speed as to set up a back pressure of 90 volts. If the internal resistance of the motor is 0·1 ohm, determine the E.M.F. of supply.

(12) A series motor is supplied with current at a constant pressure of 210 volts. Determine how the current will be affected if the load on the motor is changed so that the rate at which it works is increased 20 per cent. if the resistance remains constant and the back pressure for the first load was 60 volts.

(13) The electrical efficiency of a motor worked from constant pressure (E) mains is η. If the total resistance is R ohms, prove that the energy wasted as heat is $\frac{E^2(1-\eta)^2}{R}$ joules per second.

(14) A pulley makes n revolutions per minute; determine its angular speed, i.e. radians per second.

(15) A motor delivers 2·5 H.P. at its pulley, which is 1 foot 6 inches in diameter, when running at 900 revolutions per minute; determine the tangential force exerted at the rim of the pulley.

(16) What will be the tangential force if the pulley of the motor in (15) is replaced by another pulley 2 feet in diameter when running at the same speed and delivering 2·5 H.P.?

(17) A machine is driven from a pulley 4 feet in diameter by means of a belt. If the difference between the tensions of the two sides of the belt is 50 pounds weight, and the pulley makes 600 revolutions per minute, determine the horse-power transmitted by the belt.

(18) Determine the torque and tangential force at the pulley in No. 17.

(19) Determine the difference between the tensions of the two sides of a belt when 25 H.P. is transmitted by the belt if the pulley is 5 feet in diameter and makes 120 revolutions per minute.

(20) A certain 2-pole motor gives out 5 H.P. when supplied with 25 amperes. Given that the magnetic flux is 2500000, and that it makes 750 revolutions per minute, determine the counter-E.M.F., the number of armature conductors, and the torque.

(21) A 4-pole shunt motor working with an efficiency of 96 per cent. gives out 100 B.H.P. when run from 500-volt mains. N = 15000000, and number of armature conductors = 200, the resistance of which is 0·05 ohm. Determine counter-E.M.F., speed, and torque.

(22) A 4-pole motor armature, 24 inches diameter, series-wound with 202 external conductors, takes 300 amperes at 210 volts. Flux of lines from one pole 9000000 C.G.S. measure (1500 lines English measure). Assume field magnets to be separately excited and armature to work with 95 per cent. efficiency. Find (a) speed,

(*b*) horse-power, (*c*) tangential pull on armature conductors. (C. and G., 93.)

Section II. Electric Traction Problems.

§ 66. Electric Traction Problems. In dealing with electric traction problems we are mostly concerned with the *mechanical pull*, or its equivalent in *rotating power* applied to the axle, required to keep a tramcar moving at a uniform speed on rails, and it is obvious that the factors affecting the power expended are:—

(1) The weight of the car and its load;

(2) The resistance to motion; and

(3) The gradient, or the inclination of the line.

From the laws of motion it is well known that when a body is in motion on a level surface the force required to keep it moving at a uniform rate is determined by the magnitude of the resistance to motion. This is usually given in pounds weight per ton of the mass being moved, and is equal to the *tractive effort* required per ton when the conditions of equilibrium are satisfied; the resistance to motion, or *coefficient of traction*, is denoted by the letter f_t, and in magnitude depends upon the condition of the track, rolling friction of the wheels on the rails, friction of axles, air resistance, and the existence of curves.

According to D. K. Clark, the resistance to motion may be found in the case of steam railways as follows:—

For locomotive and train, resistance in pounds per ton is

$$f_t = \frac{m^2}{171} + 8 = \frac{(\text{speed in miles per hour})^2}{171} + 8;$$

for train only

$$f_t = \frac{m^2}{240} + 6 = \frac{(\text{speed in miles per hour})^2}{240} + 6,$$

in which m is the velocity in miles per hour.

Mr. C. E. Wolffe, B.Sc., also gives the following formula for the tractive resistance on steam railways:—

$$f_t = 3\left\{\frac{m+12}{m+2}\right\} + \frac{m^2}{300}$$

For street traffic (cars on rails) the value of f_t varies considerably, and it is usual to take as an average value of f_t, 30 pounds per ton, and with respect to the effect of curves Reckenzaun has observed that with curves of 600 inches radius the coefficient of traction is doubled, and is trebled for curves of 400 inches radius.

To find the power corresponding to the motion of a car or train at a uniform speed on a level straight track it is obviously only

necessary to multiply the actual pull exerted by the distance moved through in unit time; or if p = the total pull in pounds exerted, and v = speed, which should be given in feet per minute, we have

rate at which work is done $= p \times v$ foot-pounds per minute,

but $p = W \times f_t$

where W = total weight of the car in tons.

$\therefore$ if P = useful expenditure of power (foot-pounds per minute)

$$P = W \times f_t \times v \text{ foot-pounds per minute,}$$

but $v = \frac{1760 \times 3}{60} m$ (m = speed in miles per hour)

$$= 88\,m,$$

$$\therefore \quad P = 88\,W \times f_t \times m$$

and
$$\text{H.P. output} = \frac{W f_t v}{33000} = \frac{88 W f_t m}{33000} = \frac{8 W f_t m}{3000}.$$

If η_p be the percentage efficiency of the motor and gearing, the power which must be supplied to propel a tramcar at a uniform rate on level rails is found as follows:—

$$\text{Output} = \frac{\eta_p}{100} \times \text{power absorbed.}$$

$$\therefore \quad \text{power absorbed} = \frac{100}{\eta_p} \times \text{output}$$

$$= \frac{100}{\eta_p}\left\{\frac{W f_t v}{33000}\right\} = \frac{100}{\eta_p}\left\{\frac{88 W f_t m}{33000}\right\}$$

$$= \frac{W f_t v}{330\,\eta_p} = \frac{8 W f_t m}{30\,\eta_p}.$$

When the track is not level the effect of gravity has to be taken into consideration, and the power required for propelling a car up or down an incline may conveniently be divided into two components:—

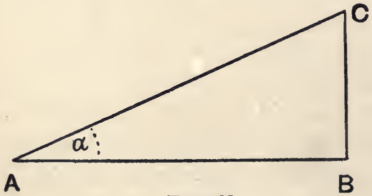


FIG. 62.

(1) The portion overcoming the tractional resistance experienced when running on the level, and

(2) The portion which is equivalent to the work done in lifting the car through a vertical distance against gravity.

To determine the effect upon the tractive effort in the case of gradients, let AC (Fig. 62) be the slope which makes an angle of inclination, α, with the horizontal AB, and it is obvious that the

work done in propelling a car up the slope from A to C will be the same as if the car were taken a distance on the level equal to AC and then lifted vertically through a distance equal to BC. Now the ratio of the lengths BC to AC is known as the *gradient*, the numerical value of which is usually expressed by giving the length of AC for which BC is unity.

Thus a gradient of 1 in 50 means a vertical rise of 1 foot for each 50 feet along the slope. But since a is the inclination of the track, and $\frac{BC}{AC} = \sin a$, the gradient may also be expressed as the sine of the angle of inclination, and the supplementary tractive effort due to the slope is clearly proportional to $\sin a$.

Now the resistance to motion vertically is equal to the weight of the body moved in pounds, but the distance through which it is moved vertically is $^1/_n$th of the distance along AC in the same time, where $^1/_n$ or $\sin a$ is the gradient. Consequently the total tractive effort required for the propulsion of cars in the case of gradients is

$$p = W\left(f_t \pm \frac{2240}{n}\right) = W\left(f_t \pm 2240 \sin a\right),$$

the + and − signs being given, since gravity may oppose or assist the motion, the + sign being used for the *up* grades and the − sign for the *down* grades, the weight of a car being an effort which assists motion in the case of a descending gradient.

Therefore, when the speed is v feet per minute, the rate of working is

$$P = Wv\left(f_t \pm \frac{2240}{n}\right) \text{ foot-pounds per minute}$$

$$= 88\,Wm\left(f_t \pm \frac{2240}{n}\right) = 88\,Wm\left(f_t \pm 2240 \sin a\right) \text{ foot-pounds per minute.}$$

$$\therefore \text{ H.P.} = \frac{Wv}{33000}\left(f_t \pm \frac{2240}{n}\right) = \frac{88\,Wm}{33000}\left(f_t \pm \frac{2240}{n}\right)$$

$$= Wm\left(\frac{f_t}{375} \pm \frac{6}{n}\right) \text{ approximately}$$

$$= Wm\left(\frac{f_t}{375} \pm 6 \sin a\right) \text{ approximately,}$$

in which m is the speed in miles per hour. If η_p is the commercial efficiency and E the pressure of supply, then the current supplied will be

$$C = \frac{100\,Wm\left(\frac{f_t}{375} \pm 6 \sin a\right) 746}{\eta_p E}.$$

When getting up speed the forces which come in operation are frictional resistance, weight, inertia, and gradient, and the tractive effort required is much greater than that corresponding to the uniform motion of the car, and it is necessary to communicate to the mass of the car an additional amount of energy to enable it to attain a definite acceleration in a certain time. In other words, the motor takes more current to enable it to exert more torque, the magnitude of the extra pull being dependent upon the time taken to attain the desired velocity, and the magnitude of the resistance due to the inertia of the car. According to the laws of motion an acceleration a results when a force f is applied to a mass M, such that

$$f = \mathrm{M}a \quad \text{or} \quad a = \frac{f}{\mathrm{M}}.$$

When starting a car of mass M from rest a force equal to f and termed *inertia resistance* has to be overcome, the measure of which is the product of the mass into the acceleration produced; consequently this is the addition to the ordinary resistance to motion which is called into operation during the time the motion is being increased. On the other hand, this inertia resistance becomes an effort in the direction of motion when stopping a car, and must be absorbed by a brake. Furthermore, a body starting from rest acquires a velocity v when the body has traversed a distance s feet, such that

$$v^2 = 2sa = 2s\frac{f}{\mathrm{M}},$$

from which

$$fs = \tfrac{1}{2}\mathrm{M}v^2.$$

Now, the product fs is the work in foot-pounds done whilst the force f acts through s feet, and this is the amount of energy stored as kinetic energy in the moving body. If the weight of the body is W, then

$$\mathrm{M} = \frac{\mathrm{W}}{g},$$

where g is the acceleration due to gravity, and which may be taken as 32 feet per sec. per sec., since the weight of a body = its mass × the acceleration of gravity.

Consequently when a car of weight W tons starting from rest attains a speed of m miles per hour, the energy stored is

$$k = \tfrac{1}{2}\mathrm{M}v^2 = \tfrac{1}{2} \times \frac{\mathrm{W} \times 2240}{g} \times \left(\frac{m \times 5280}{60 \times 60}\right)^2$$

$$= \frac{\mathrm{W} \times 2240 \times m^2 \times 88^2}{2 \times 32 \times 60^2} \text{ foot-pounds}$$

$$= 75{\cdot}3\,\mathrm{W}m^2 \text{ foot-pounds.}$$

The distance s traversed if this amount of energy is acquired in t minutes is given by the product of the mean velocity and the time, or

$$s = \tfrac{1}{2}(m \times 88) \times t = 44mt,$$

and the average extra pull p_i called into operation by the inertia resistance is such that

$$k = p_i \times s$$

$$\therefore \quad p_i = \frac{k}{s} = \frac{75{\cdot}3\,\mathrm{W}m^2}{44mt}$$

$$= 1{\cdot}711\,\frac{\mathrm{W}m}{t}$$

therefore the total tractive effort required to start a car from rest in t minutes so as to acquire a speed of m miles per hour is

$$p = \mathrm{W}\left\{f_t \pm \frac{2240}{n} + 1{\cdot}711\,\frac{m}{t}\right\}$$

$$= \mathrm{W}\left\{f_t \pm 2240 \sin a + 1{\cdot}711\,\frac{m}{t}\right\}.$$

If p_a = average pull per ampere required to give to a car of weight W tons a speed of m miles per hour in t minutes, then the average current will be such that

$$p_a\mathrm{C} = p$$

and

$$\mathrm{C} = \frac{\mathrm{W}\left\{f_t \pm 2240 \sin a + 1{\cdot}711\,\frac{m}{t}\right\}}{p_a}$$

Of course the current just at starting will be much greater than this; but as t is usually small, this may be taken as the average value of the current. The useful expenditure of power during the time t is:

$$\mathrm{P} = 88\,\mathrm{W}\,m\left\{f_t \pm 2240 \sin a + 1{\cdot}711\,\frac{m}{t}\right\} \text{ foot-pounds per minute}$$

$$= \frac{88}{33000}\,\mathrm{W}m\left\{f_t \pm 2240 \sin a + 1{\cdot}711\,\frac{m}{t}\right\} \text{ H.P.}$$

$$= \frac{8}{3000}\,\mathrm{W}m\left\{f_t \pm 2240 \sin a + 1{\cdot}711\,\frac{m}{t}\right\} \text{ H.P.}$$

To reduce the speed or stop a moving car some form of brake is used, the function of which is to absorb $\tfrac{1}{2}\mathrm{M}v^2$ foot-pounds of kinetic energy, and this is effected by the conversion of the kinetic energy into its equivalent amount of thermal energy, either directly or indirectly. If, when stopping a car which is going down an incline, the current is cut off, the tractive effort due to gravity must be taken

into account, in addition to the $\frac{1}{2}Mv^2$ units which has to be absorbed. If the time taken to bring a car to rest, or the distance traversed in doing this, be required, they may be obtained by applying one or more of the following well-known relationships:

$$v = at,$$
$$s = \tfrac{1}{2}at^2,$$
$$v^2 = 2as.$$

Worked Examples. (1) Determine the horse-power required to keep a car weighing 10 tons running on the level at $7\frac{1}{2}$ miles per hour if the resistance to traction is 30 pounds per ton. If the efficiency of the motor and driving arrangements is 75 per cent., find the watts supplied.

$$\text{Resistance to overcome} = p = Wf_t$$
$$= 10 \times 30 = 300 \text{ pounds.}$$
$$\text{Speed (feet per minute)} = \frac{7\frac{1}{2} \times 1760 \times 3}{60}$$
$$= 660 \text{ feet per minute.}$$
$$\therefore \quad \text{Work done per minute} = 660 \times 300$$
$$= 198000 \text{ foot-pounds}$$

and
$$\text{H. P.} = \frac{198000}{33000} = 6 \text{ H.P.}$$
$$\text{Output in watts} = 6 \times 746$$
$$\therefore \quad \text{Watts supplied} = \frac{6 \times 746 \times 100}{75}$$
$$= 5968 \text{ watts.}$$

(2) An electric tramcar of 12 tons rolling weight runs up an incline of 1 in 100 at a speed of 8 miles an hour. If the resistance to traction on the level is 30 pounds per ton, and the E.M.F. of supply is 500 volts, determine (*a*) the horse-power which must be supplied, (*b*) the current, and (*c*) the pull per ampere, given that the efficiency of the propelling mechanism is 60 per cent.

(*a*) Resistance to traction on the level $= 12 \times 30 = 360$ pounds.

$$\text{Speed} = \frac{8 \times 5280}{60} = 704 \text{ feet per minute.}$$
$$\text{Corresponding work} = 360 \times 704 \text{ foot-pounds per minute.}$$
$$\text{Velocity vertically} = \frac{1}{100} \times 704 \text{ feet per minute.}$$
$$\text{Work done vertically} = 12 \times 2240 \times 7{\cdot}04 \text{ foot-pounds per minute.}$$
$$\text{Total work done} = 704\,(360 + 12 \times 22{\cdot}4) \text{ foot-pounds per minute}$$
$$= 442675{\cdot}2 \text{ foot-pounds per minute.}$$
$$\therefore \quad \text{H.P.} = 13{\cdot}4144 \text{ H.P.}$$

Since this is 60 per cent. of that supplied

$$\therefore \quad \text{H.P. supplied} = \frac{100}{60} \times 13{\cdot}4144$$
$$= 22{\cdot}357 \text{ H.P.}$$

(b) $$\text{Watts supplied} = 22{\cdot}357 \times 746$$
$$= C \times 500.$$

$$\therefore \quad \text{Current required} = C = \frac{22{\cdot}357 \times 746}{500} \text{ amperes}$$
$$= 33{\cdot}5366 \text{ amperes.}$$

(c) $$\text{Resistance to traction on level} = 12 \times 30 = 360 \text{ pounds.}$$

$$\text{Resistance to traction vertically} = \frac{12 \times 2240}{100} = 268{\cdot}8 \text{ pounds.}$$

$$\text{Total tractive effort} = 628{\cdot}8 \text{ pounds.}$$

$$\therefore \quad \text{Current} \times \text{pull per ampere} = 628{\cdot}8 \text{ pounds}$$

and $$\text{pull per ampere} = \frac{628{\cdot}8}{33{\cdot}5366} \text{ pounds}$$
$$= 18{\cdot}85 \text{ pounds.}$$

(3) A tramcar weighing 12 tons takes 15 seconds to attain a speed of 8 miles an hour on a gradient of 1 in 100; if the resistance to traction is 30 pounds per ton, determine the horse-power required if the efficiency is 70 per cent.

The power required to overcome inertia resistance, frictional resistance, and gravity may be determined separately, the sum of which gives the total power required. A speed of 8 miles an hour is $8 \times \frac{88}{60}$ feet per second.

(a) $$\text{Kinetic energy} = \tfrac{1}{2} M v^2$$
$$= \frac{12 \times 2240}{2 \times 32} \times \left(8 \times \frac{88}{60}\right)^2 \text{ foot-pounds}$$
$$= 57822 \text{ foot-pounds.}$$

This amount of work done in 15 seconds corresponds to an expenditure of power as follows:

$$\text{H.P.} = \frac{57822}{15} \times \frac{1}{550} = 7 \text{ H.P.}$$

(b) Frictional resistance requires

$$\text{H.P.} = \frac{12 \times 88 \times 8}{33000} \times 30$$
$$= 7{\cdot}68.$$

(c) Gravity requires

$$\text{H.P.} = \frac{12 \times 2240}{100} \times 8 \times 88$$
$$= 5{\cdot}73.$$

$$\text{Total H.P.} = 7 + 7{\cdot}68 + 5{\cdot}73$$
$$= 20\ 41 \text{ H.P.}$$

$$\therefore \quad \text{H.P. required} = \frac{100}{70} \times 20{\cdot}41$$
$$= 29{\cdot}16 \text{ H.P.}$$

(4) A car weighing 10 tons when loaded gets out of control down an incline of 1·5 in 100, and moves at the rate of 20 miles an hour, and on the application of the emergency brake it is pulled up in a distance of 20 yards. What is the energy dissipated, the average brake-power, the retardation, and the time taken to pull up?

(*a*) Kinetic energy of moving car $= \frac{1}{2} M v^2$

$$= \frac{1}{2} \times \frac{10 \times 2240}{32} \times \left(\frac{20 \times 5280}{60 \times 60}\right)^2$$

$$= 301155\tfrac{5}{9} \text{ foot-pounds.}$$

$$\text{Work done by gravity} = \frac{10 \times 2240}{n} \times \frac{20 \times 5280}{60 \times 60}$$

$$= 10 \times 2240 \times \frac{1 \cdot 5}{100} \times \frac{88}{3} \text{ foot-pounds}$$

$$= 9856 \text{ foot-pounds.}$$

$$\therefore \quad \text{Energy dissipated} = 301155\tfrac{5}{9} + 9856$$

$$= 311011\tfrac{5}{9} \text{ foot-pounds.}$$

(*b*) $$\text{Average braking-power} = \frac{311011\frac{5}{9}}{\frac{88}{3}}$$

$$= 10602 \cdot 66 \text{ pound-weight.}$$

(*c*) Let a = retardation or negative acceleration,

then $$f = M \times a$$

and $$a = \frac{10602 \cdot 65}{\frac{10 \times 2240}{32}}$$

$$= \frac{10602 \cdot 66}{700} = 15 \cdot 146 \text{ feet per sec. per sec.}$$

(*d*) Since $$v = a\,t$$

$$\therefore \quad t = \frac{v}{a} = \frac{\frac{88}{3}}{15 \cdot 146}$$

$$= 1 \cdot 93 \text{ seconds.}$$

EXERCISES IX B.

(1) The total weight of a tramcar is 5 tons. Determine the speed in miles per hour at which the car runs on the level when the back E.M.F. of the motor is 90 volts and the current taken 25 amperes, the resistance to motion being 30 pounds per ton, the frictional losses, &c., being neglected.

(2) Determine the current required to run a car of 10 tons weight at 10 miles per hour on the level if the resistance to motion is 30 pounds per ton and the E.M.F. of supply is 500 volts, the efficiency of the motor and gearing being 75 per cent.

(3) The E.M.F. of supply of an electric traction circuit is 500 volts, and the total resistance of line and car is 5 ohms. Determine the speed of a car weighing 10 tons on the level when the current is 20 and 25 amperes respectively, if the resistance to motion is 30 pounds per ton and the gearing absorbs 16 per cent. of the power supplied.

(4) An electric tramcar of 10 tons rolling weight is required to run

up an incline of 1 in 60 at a uniform speed of 7 miles per hour. The resistance to traction on a level line is 30 pounds per ton, and the total or combined efficiency of the propelling mechanism (ratio of mechanical power utilized to the electrical power supplied) is 57 per cent. What is the electrical horse-power which must be supplied? And what is the current required if the supply is at a constant pressure of 300 volts? (C. and G.)

(5) An electric tramcar, total weight 10 tons, runs upwards on an incline of 1 in 100. The motor and gear are such that for every ampere passing through the armature there is exerted a tractive effort of 10 pounds. The resistance to traction on the level is 30 pounds per ton. How many amperes are required to propel the car up the incline? Give the horse-power corresponding to a speed of 4 miles per hour. (C. and G.)

(6) An electric tramcar, total weight 10 tons, runs on the level at a speed of 6 miles per hour. If resistance to motion be 30 pounds per ton, determine the horse-power corresponding to this speed if the commercial efficiency is 60 per cent. Also determine the work done if this speed is reached after running 50 feet, and the average acceleration during this period.

(7) Determine the tractive effort per ton required to produce an acceleration of one mile an hour per second on a level track, neglecting frictional resistance.

(8) A car of weight 12 tons is running on a horizontal track at the rate of 10 miles per hour; resistance to motion is 30 pounds per ton. If the current is cut off, find (*a*) the time that elapses before the car comes to rest, and (*b*) the distance described in this time.

(9) If the car, exercise 8, be ascending an incline of 1 in 100, find the time and distance to bring the car to rest when the current is cut off.

(10) An electric tramcar 10 tons weight is running at the rate of 10 miles an hour down an incline of 1 in 80; determine the resistance required to stop it in 220 yards if the power be cut off.

(11) What torque in inch-pounds must be exerted by each of the two motors on a tramcar weighing 10 tons, so that on a level track they may produce an acceleration of 1·2 ft. per sec. per sec.? Tractive force 30 pounds per ton; gearing ratio 4·8, wheels 33 inches diameter; efficiency of gearing 85 per cent. (C. and G.)

(12) A tramcar of weight 10 tons is running at the rate of 10 miles per hour on the level, and it is found that it takes 5·2 seconds for a brake to bring it to rest after the current is cut off. Determine the force exerted, assuming it to be uniform, the energy dissipated, and the distance travelled in pulling up.

(13) A car weighs 10 tons; what is its mass in engineers' units? It is drawn by the pull P lb. varying in the following way, t being seconds from the time of starting.

P	1020	980	882	720	702	650	713	722	805
t	0	2	5	8	10	13	16	19	22

The retarding force of friction is constant, and equal to 410 lb. Plot P − 410 and the time t, and find the *time average* of this excess force. What does this represent when it is multiplied by 22 seconds? What is the speed of the car at the time 22 seconds from rest? Tabulate values of speed and time, and draw a curve showing speed and time.

CHAPTER X

EFFICIENCIES AND DYNAMO CALCULATIONS

§ **67. Efficiencies.** As with all kinds of machinery, the *output* of electric machines is always less than the *intake*, and, adopting the principle introduced by Rankine, various ratios may be formed between the useful energy given out, the total energy produced, and the total energy supplied, to which the name *efficiency* has been given. And since the total energy developed both by dynamos and motors may be divided into two parts—i e. (1) the useful, and (2) the useless—it is obvious that three important ratios may be formed, known respectively as

(1) the electrical efficiency;

(2) the gross efficiency, mechanical efficiency or efficiency of conversion; and

(3) the commercial or net efficiency.

These are defined as follows:

$$\text{Electrical efficiency} = \eta_e = \frac{\text{useful energy given out}}{\text{total energy developed}}$$

$$= \frac{\text{external activity}}{\text{internal activity} + \text{external activity}}$$

$$\text{Gross efficiency} = \eta_g = \frac{\text{total energy developed}}{\text{total energy imparted to machine}}$$

$$\text{Commercial efficiency} = \eta_c = \frac{\text{useful energy given out}}{\text{total energy imparted to machine}}$$

$$= \frac{\text{output}}{\text{input}} = \frac{\text{input} - \text{losses}}{\text{input}}.$$

If $\quad \mathrm{W}$ = total energy supplied

w = useful energy given out

w = energy wasted in the machine

η_e, η_g, and η_c may be stated algebraically as follows:

$$\eta_e = \frac{\mathrm{w}}{\mathrm{w} + w} \quad ; \quad \eta_g = \frac{\mathrm{w} + w}{\mathrm{W}}$$

$$\eta_c = \frac{\mathrm{w}}{\mathrm{W}}.$$

A moment's consideration of the above statements makes it clear

that an efficiency must in all cases be represented by a quantity less than unity, being expressed as a ratio, the value of which is a proper fraction. For many purposes it is convenient and usual to state the efficiency of a machine as a percentage. Thus if $\eta = \frac{4}{5} = 0{\cdot}8$, the percentage efficiency is $0{\cdot}8 \times 100$ or 80 per cent. And although it is practically impossible to attain an efficiency equal to unity, good workmanship and design have done much to render both dynamos and motors highly efficient by diminishing the following individual losses composing w:—

Electrical losses, $C_a{}^2 r_a$, those due to armature resistance, $C^2_{se} r_{se}$ or $C_{sh}{}^2 r_{sh}$ due to resistance of the field-magnet coils, eddy or Foucault currents in the cores of the armature and field-magnets.

Magnetic losses due to hysteresis.

Mechanical losses, friction at the bearings, friction between the brushes and commutator, and air resistance encountered by the rotating armature or 'windage' as it is termed.

The following values of the efficiencies of dynamos and motors, series and shunt-wound, are given as an exercise.

Electrical Efficiencies.

Series dynamo, $\eta_e = \dfrac{Ce}{CE} = \dfrac{e}{E}$

$$= \frac{R}{R + r_a + r_{se}}$$

Shunt dynamo,

$$\eta_e = \frac{C^2 R}{C^2 R + C_a{}^2 r_a + C_{sh}{}^2 r_{sh}}$$

Series motor, $\eta_e = \dfrac{\epsilon}{E}$

Shunt motor, $\eta_e = \dfrac{C_a \epsilon}{E\left(C_a + \dfrac{E}{r_{sh}}\right)}$

$$= \frac{C_a \epsilon}{E(C_a + C_{sh})}$$

Commercial Efficiencies.

Series dynamo, $\eta_c = \dfrac{Ce}{W}$

Shunt dynamo, $\eta_c = \dfrac{Ce}{W}$

Series motor, $\eta_c = \dfrac{C\epsilon - w}{CE}$

$$= \frac{\epsilon}{E} - \frac{w}{CE}$$

Shunt motor, $\eta_c = \dfrac{C_a \epsilon - w}{E\left(C_a + \dfrac{E}{r_{sh}}\right)}$

§ 68. Determination of Speed. If a series dynamo be run at a speed of n_1 revolutions per minute, then the E.M.F., E, generated is

$$E = e + C(r_a + r_{se})$$

where e = terminal pressure. But if the same machine be supplied with the same current C, then

$$E_1 = \epsilon + C(r_a + r_{se}),$$

where $\quad E_1$ = pressure of supply,

and $\quad \epsilon$ = counter-E.M.F. of the motor,

and the E.M.F. induced is

$$\epsilon = E_1 - C(r_a + r_{se}),$$

and let us suppose that the speed at which this back pressure is generated is n_2 revolutions per minute, then since the magnetic fields may be taken to have the same strength both for the dynamo and the motor the voltages induced will be proportional to speeds, i.e.

$$\frac{E}{\epsilon} = \frac{n_1}{n_2} \quad \text{and} \quad n_2 = n_1 \times \frac{\epsilon}{E}$$

$$= \frac{E_1 - C(r_a + r_{se})}{e + C(r_a + r_{se})} \times n_1.$$

Worked Examples. (1) A motor giving out $1\frac{1}{4}$ H.P. is supplied with 10 amperes at 120 volts. Determine the electrical and commercial efficiencies if the resistance of the motor is 2 ohms.

$$\text{Energy wasted in watts} = C^2 R = 100 \times 2$$
$$= 200 \text{ watts}$$

E C

$$\text{Energy supplied} = C E = 10 \times 120$$
$$= 1200$$

E Cr

$$\therefore \quad \text{Electrical efficiency} = \eta_e = \frac{1200 - 200}{1200} = \frac{5}{6}$$
$$= 83{\cdot}3 \text{ per cent.}$$

$$\text{Commercial efficiency} = \eta_c = \frac{1\frac{1}{4}\text{ H.P.}}{1200} = \frac{373}{480}$$
$$= 77{\cdot}7 \text{ per cent.}$$

(2) A series dynamo produces 104 amperes at a terminal P.D. of 220 volts at a speed of 961 revolutions per minute. $r_a = 0{\cdot}115$ ohm and $r_{se} = {\cdot}0125$ ohm. (*a*) Determine the speed at which the machine will run as a motor if supplied with 104 amperes at 220 volts. (*b*) Determine the electrical efficiencies of the machine as a generator and motor. (*c*) Determine the E.M.F. of supply so that the motor will run at 961 revolutions per minute if the current be 104 amperes.

(*a*) As a dynamo, pressure generated at 961 r.p.m. is

$$E = 220 + 104\,({\cdot}115 + {\cdot}0125)$$
$$= 233{\cdot}25 \text{ volts.}$$

As a motor, counter-E.M.F. set up is

$$E = 220 - 104\,({\cdot}115 + {\cdot}0125)$$
$$= 206{\cdot}75 \text{ volts.}$$

But $$\frac{n_2}{n_1} = \frac{\epsilon}{E} \quad \text{and } n_2 = \frac{206{\cdot}75}{233{\cdot}25} \times 961$$
$$= 850 \text{ r.p.m.}$$

(*b*) Electrical efficiency as dynamo

$$\eta_e = \frac{Ce}{CE} = \frac{e}{E} = \frac{220}{233{\cdot}25}$$
$$= 94{\cdot}3 \text{ per cent.}$$

Electrical efficiency as motor

$$\eta_e = \frac{\epsilon}{E} = \frac{206{\cdot}75}{220}$$
$$= 93{\cdot}98 \text{ per cent.}$$

(*c*) If the motor runs at 961 r.p.m. and the current is 104 amperes, the counter-E.M.F. of motor is $\epsilon = E =$ pressure generated by dynamo at 961 r.p.m.

$$\therefore \quad \epsilon = 233{\cdot}25 \text{ volts,}$$

but
$$\text{volts lost} = C\,(r_a + r_{se}) = 104\,({\cdot}115 + {\cdot}0125)$$
$$= 13{\cdot}25 \text{ volts.}$$

$$\therefore \quad \text{required pressure of supply} = 233{\cdot}25 + 13{\cdot}25$$
$$= 246{\cdot}5 \text{ volts.}$$

(3) A shunt-wound dynamo running at 1200 revolutions per minute gives 38 amperes at a terminal P.D. of 130 volts. The armature resistance is 0·2 ohm, and the resistance of the shunt winding is 50 ohms. (*a*) What speed will it run at as a motor, when supplied with current at 130 volts, if it gives out 4·25 H.P., and the external and frictional losses amount to 0·75 H.P.? (*b*) Determine the electrical and commercial efficiencies.

(*a*)
$$\text{Total power} = 4{\cdot}25 + 0{\cdot}75 = 5 \text{ H.P.}$$
$$= 5 \times 746 = 3730 \text{ watts}$$
$$= C_a \times \epsilon$$
$$\therefore \quad \epsilon = \frac{3730}{C_a} \text{ volts,}$$

but
$$\text{drop in volts in the armature} = C_a \times r_a$$
$$= 0{\cdot}2\,C_a \text{ volts}$$
$$\therefore \quad \text{counter-E.M.F.} = \epsilon = E - 0{\cdot}2\,C_a$$
$$= 130 - 0{\cdot}2\,C_a$$
$$\therefore \quad 130 - 0{\cdot}2\,C_a = \frac{3730}{C_a}$$

and
$$C_a^2 - 650\,C_a + 18650 = 0,$$

from which
$$C_a = 30 \text{ (approx.)}$$

and
$$\text{drop in volts in the armature} = 0{\cdot}2 \times 30$$
$$= 6 \text{ volts.}$$
$$\therefore \quad \text{counter-E.M.F.} = \epsilon = 130 - 6 = 124 \text{ volts.}$$

Therefore, at a certain speed, a counter-E.M.F. of 124 volts is produced; but when running as a dynamo with the same field excitation, the E.M.F. induced at 1200 r.p.m. is $(130 + C_a r_a)$ volts. Now, when running as a dynamo

$$C_a = 38 + C_{sh} = 38 + \frac{130}{r_{sh}} = 38 + \frac{130}{50}$$
$$= 38 + 2{\cdot}6 = 40{\cdot}6 \text{ amperes}$$

$\therefore$ E.M.F. induced at 1200 r.p.m. is

$$130 + C_a r_a = 130 + 40{\cdot}6 \times {\cdot}2$$
$$= 138{\cdot}12 \text{ volts.}$$

But
$$\frac{n_2}{n_1} = \frac{\epsilon}{E} = \frac{E_1 - C_a \times r_a}{e + C_a r_a}$$
$$= \frac{130 - 6}{130 + 8 \cdot 12} = \frac{124}{138 \cdot 12}$$

and
$$n_2 = \frac{124}{138 \cdot 12} \times 1200$$
$$= 1077 \text{ r.p.m.}$$

Note.—Current supplied $= C_a + C_{sh}$
$$= 30 + 2 \cdot 6 = 32 \cdot 6 \text{ amperes.}$$

(*b*)
$$\text{Electrical efficiency} = \eta_e = \frac{C_a \epsilon}{E\ (C_a + C_{sh})}$$
$$= \frac{30 \times 124}{130 \times 32 \cdot 6}$$
$$= 87 \cdot 7 \text{ per cent.}$$
$$\text{Commercial efficiency} = \eta_c = \frac{C_a \epsilon - w}{E\ (C_a + C_{sh})}$$

where
$$w = 0 \cdot 75 \text{ H.P.} = 559 \cdot 5 \text{ watts,}$$
$$\therefore \quad \eta_c = \frac{3730 - 559 \cdot 5}{130 \times 32 \cdot 6}$$
$$= 74 \cdot 8 \text{ per cent.}$$

EXERCISES X.

(1) A series motor is supplied with 60 amperes 120 volts, and runs at such a speed that its counter-E.M.F. is 115 volts. Determine the output in H.P. and its commercial efficiency if the frictional losses are 0·4 H.P.

(2) Two exactly similar series machines are used for transmitting power, one acting as a generator and the other as a motor. At a certain speed the dynamo generates 110 volts, and the motor is required to give a B.H.P. of 8 H.P. The internal resistance of each is 0·04 ohm, and the connecting leads have a resistance of 0·025 ohm. Assume that the losses due to friction in each machine are 400 watts. If the current passing is 64 amperes, determine the C.E.M.F. of the motor, the electrical efficiency of the motor, and the actual efficiency of the plant.

(3) A series dynamo has an armature resistance of 0·4 ohm and a field resistance of 0·6 ohm, and gives 120 terminal volts at a given speed, and a current of 10 amperes. Work out its electrical efficiency as a dynamo and as a motor at that speed and current. (C. and G.)

(4) A shunt machine, in which $r_a = 3$ ohms and $r_{sh} = 625$ ohms, gives a terminal P.D. of 500 volts when running at 1200 r.p.m. as a dynamo and supplying 15 amperes. Determine the speed and the brake horse-power of the machine when working as a motor and receiving 15 amperes at 500 volts, if 0·5 H.P. is lost in various frictions.

CHAPTER XI

SYSTEMS OF ELECTRICAL UNITS.

§ **69. Introduction.** The study of physical, and especially electrical, science is undoubtedly a most attractive one, and the pleasure attending experimental work has enticed many workers to this branch of study. The early development of any branch of applied science depends largely upon experimental work, but the experimental determination of the characteristic properties and effects is but the first step in the investigation of any physical phenomenon, for after these have been isolated it becomes necessary to make relative measurements and to compare the magnitudes of the quantities discovered experimentally. We are now assured that nothing occurs in the physical world but what is in accordance with natural law, and the relationships expressed in terms of a law are the foundations upon which applied science is developed. To-day much of the applied electrical work has left the domain of experimental science and become pre-eminently quantitative, consequently it is necessary to utilize units and standards. In the previous chapters various physical actions and effects have been expressed as functions of dynamical force, and, as a matter of fact, the magnitudes of electrical and magnetic quantities have been defined by their mechanical properties. We, however, find at the threshold of physical science that there are three primary conceptions or quantities which cannot be defined in terms of anything simpler than themselves; these are length, mass and time. Moreover, the measure of a physical effect is dependent upon a unit of force, consequently all the electrical and magnetic quantities with which we have to deal are functions of the three fundamental quantities, length, mass and time.

There are two essential factors, i.e. (1) the quantitative character of physical science, and (2) the dependence of the development of applied science upon the interchange of exact thought, observation and measurement, which render it necessary that all the results obtained in practice should be brought into conformity by means of a universal system of units and standards. The primary object, for instance, of every system of measurement is the determination of the value of one magnitude in terms of the unit quantity of the same

kind, and before we can do this we must be able to reduce all kinds of physical quantities to one common scale of comparison; it is therefore absolutely necessary to select suitable units and standards by means of which physical quantities of any magnitude may be expressed and compared. One has only to refer to papers on electrical and magnetic subjects published previously to 1850, to be forcibly impressed with the waste of brain power which must result from the absence of a universal system of units, and with the necessity for specifying the fundamental units used, if one wishes to compare physical results obtained by different people. At this early period experimental results in electricity and magnetism, which were published, were simply relative, and were expressed in terms of units and standards individually and arbitrarily chosen, the magnitudes of which were chosen and even frequently changed (in the same article) to suit the convenience and whim of the investigator. The Daniell's cell, for instance, was often used as a standard of electromotive-force, whilst a certain length of wire of a certain gauge was frequently adopted as a standard of resistance. Werner Siemens was probably the first to propose a standard of resistance which could easily be reproduced, his standard being a column of mercury one metre long and one square millimetre in sectional area. In many respects, however, chaos reigned supreme, and no more interesting and instructive work in the domain of physical science could be written than one giving the complete history and development of physical units and standards, showing how uncertainty was removed and how confusion gave way to simplicity and harmony, and explaining how electricity has been made the science of exact measurement, which it undoubtedly is. As a matter of fact, the universal adoption of a system of units founded upon scientific and well-known principles is a fitting monument to the honour of a number of physicists—the master minds and greatest authorities of different nationalities—and which at the same time clearly indicates the immense advance and progress made in the development of physical and applied science.

§ 70. **Units.** All magnitudes of the same kind may be measured by the number of units which they contain, so that the numerical value or expression of a quantity involves two factors or components, (1) the particular unit of the same concrete kind as the quantity, and (2) a numeric or numerical factor stating the ratio of the quantity to its unit. Thus, if

Q_0 = the magnitude of the unit of comparison,

Q = the numerical expression of a quantity of the same kind,

then $Q = n[Q_0]$

where n is the numeric or number of units which Q contains. n may be either integral or fractional.

If the magnitude of the unit chosen be changed the value of the numeric will change inversely; thus, if the size of the unit quantity Q_0 be changed to Q_0'

then $\qquad Q = n'[Q_0']$, where n' = the new numeric,

but $\qquad Q = n[Q_0]$

therefore $\qquad n'[Q_0'] = n[Q_0]$

and $\qquad n' = n\left[\dfrac{Q_0}{Q_0'}\right]$ or $\dfrac{n'}{n} = \dfrac{[Q_0]}{[Q_0']}$

or $\qquad [Q_0] : [Q_0'] :: \dfrac{1}{n} : \dfrac{1}{n'}$.

In words, the ratio of the numerics of a given quantity is equal to the inverse ratio of the magnitudes of the unit, therefore by the definition of variation, $[Q_0] \propto \frac{1}{n}$, that is, the unit in terms of which any quantity is measured varies inversely as the measure or numeric, and vice versa. It will also be observed that $\left[\dfrac{Q_0}{Q_0'}\right]$ is the ratio of the original unit quantity to the new unit quantity; this ratio is known as the *change-ratio*, and it is that factor which, when multiplied with the original numeric, gives the new numeric required.

Now, there are two courses open for the selection of any standard; either the standard unit may be arbitrarily chosen quite independent of any other and be fixed and crystallized, as it were, in a manufactured apparatus, or it may be defined in terms of such physical properties of the universe as are at least liable to change with time and throughout space. The former is only suitable for measures which are simply relative; consequently the latter has been adopted so as to form a scientific basis for a universal system of units. It has been named the *absolute* or rational method, because on the hypothesis that it is a workable process, it would arrive at a unit which would be free from changes and decay of ordinary material, apparatus and things. The term absolute is used in contradistinction to the term special, of the relative system which was in use in the more primitive times, but this term—absolute—must not be taken to mean that the quantities expressed in absolute units are absolutely correct, but that the quantities relate to and are perfectly independent of all other arbitrary values except the definite fundamental units of length, mass and time. The absolute system utilizes the inter-relations existing between different kinds of quantities so that equations may be formed which express these relationships; thus volumes and

areas are connected with lengths, and power and acceleration are connected with force, and so on.

This brings us to the point that the arbitrary choice of unit is limited by two conditions of fundamental importance:

(1) The units must be chosen so as to simplify as much as possible the statement of the quantitative relations existing between various kinds of quantities.

(2) The units of all quantities should be invariable, unaffected by conditions of time and place, and be independent of the properties of particular bodies; in other words, they should be *absolute* and be *unconditioned.*

§ 71. **Absolute Units.** In building up an absolute system of units advantage is taken of the fact that all physical quantities may be expressed in terms of length, mass and time, consequently the units of length, mass and time are basal units and are termed *fundamental units*, because the definition of a fundamental unit involves no reference to other units, and one fundamental unit cannot be reduced from another. Units of quantities dependent upon one or more fundamental units are termed *derived* units, and it follows that the magnitude of the derived units are known when the system adopted is specified. The system of units now adopted universally is that based upon the centimetre as the unit of length, the gramme as the unit of mass, and the second as the unit of time, and is known as the C.G.S. (centimetre-gramme-second) system of units. The centimetre is approximately the billionth part of the terrestrial quadrant through Paris; more correctly, however, it is the $\frac{1}{100}$th part of the standard metre measured by Delambre and Borda, and kept at the Conservatory at Sevres. The gramme, the unit of mass, is the mass of a centimetre cube of water, and it is interesting to notice that there exists a connexion between the unit of mass and unit of length.

Before we trace the connexion between the various derived units and the fundamental units, it will be instructive to consider in a brief manner the historical aspect of the subject. The founders of our present system were undoubtedly Gauss and Weber, and of all the early physicists Gauss was probably the only one who, desirous of expressing the results of exact measurements of the distribution of terrestrial magnetism, systematically based his measurements on a unit of force which, unlike the weight of a given mass, should not depend upon the place where the test was made. Unit force he defined as that which acting on unit mass (the gramme) for unit time (the second) produced unit velocity (one metre per second). At this time the metre was recognized on the continent as the unit of length. As this definition implies, the unit of force can be expressed in terms of

the fundamental units of length, mass and time. The idea, however, was not new, for both force and work have been defined and measured in terms of these fundamental units ever since Newton founded rational dynamics. It was in 1851 that Weber laid the foundation of a complete and simply related system of units for electrical and magnetic quantities, having realized that the mutual attractions and repulsions of electrified bodies, magnets, currents and magnets, &c., are forces capable of precise measurement in dynamical units, and are independent of the constants of the instruments and the conditions of the experiments, thus indicating the possibility of expressing all the electrical and magnetic quantities in terms of the fundamental units. The outcome of Weber's idea was to express electrifications, magnetic poles, currents, &c., as numerical quantities in terms of mechanical force, and fortunately his proposals attracted the attention which they deserved, having in Lord Kelvin, then William Thomson, a most able and valuable exponent. Lord Kelvin exerted himself with his accustomed zeal, and in 1861 persuaded the British Association to take up the matter, with the result that a Committee on Electrical Units and Standards was appointed to consider the whole question, and physicists the world over are indebted to that Committee for introducing (in 1879) a system of units based on Weber's system slightly modified. It is worthy of notice that in the same year a very suggestive paper on Electrical Units by Mr. Latimer Clark and Sir C. Bright was read before the British Association, and many of the principles then first discussed were afterwards adopted as a system of practical units, and later made to form the basis of an international system of units. The most important change made by the British Association Committee was that of fixing the centimetre, instead of the metre, as the fundamental unit of length, and thus giving birth to the C.G.S. system of units. They also arranged for the supply of standard resistance coils in terms of the British Association unit of resistance.

Two years later, in 1881, an international Congress of Electricians was held at Paris, at which the C.G.S. absolute system of units received universal sanction, and a system of practical units based thereon, and in which the volt, ampere and ohm were decided upon and accepted as the recognized system for practical use.

At Chicago, in 1893, new denominations of standards based upon the above system of practical units were specified and designated as International Standards. More recently this system of practical units has been welded together and the standards definitely specified, and by an Order in Council on Electric Units and Standards issued

by the Board of Trade, in 1894, the new denominations of standards were legalized, and arrangements made whereby the standards may be verified. It will be noticed in the foregoing remarks that no names and values have been given to the practical magnetic units, although the matter was introduced at Chicago in 1893. In 1894 the American Institution of Electrical Engineers made an attempt to complete the work left unfinished the year before, by proposing to give names of illustrious electricians to the principal magnetic C.G.S. units, i.e. magnetomotive-force, reluctance, magnetic flux, and flux density. The names suggested for these units were respectively, the *Gilbert*, the *Oersted*, the *Weber*, and the *Gauss*. In 1895 at the Ipswich meeting of the British Association the question of magnetic units was also discussed, and it was suggested:

(1) That as a unit for magnetic field 10^8 C.G.S. lines be called a *Weber*. Thus the leakage of a weber per second would, according to the laws of induction, induce one volt in every turn of wire forming the coil linked with the magnetic field.

(2) That the C.G.S. unit of magnetic potential or of magnetomotive-force be called a *Gauss*. An ampere-turn would then correspond to $\frac{4\pi}{10}$ of a gauss or 1·257 gauss.

Neither the American nor the British Association suggestions have received universal sanction, but two of the four magnetic units received official endorsement at the Paris Congress of 1900, and the *Gauss* was adopted as the C.G.S. unit of flux-density, and the *Maxwell* as the C.G.S. unit of magnetic-flux.

§ 72. Elementary Principles of Mechanics. Inasmuch as the laws of electrical science are founded upon mechanical principles it will be instructive at this point to refer in a brief manner to those principles upon which the system of electrical units depends, and the most convenient starting point is the conception of force. By force we understand the sensation of muscular effort and other causes which may produce identical effects; muscular exertion impeded, for instance, impresses us very clearly with an idea of force. Everyday experience proves that if we wish to change the state of a body, either of rest or form, some force or form of energy must be applied to it, and that muscular effort or some equivalent cause may produce a movement or deformation of bodies. In other words, the effects of a force are (1) *a change of size or shape* termed *deformation* or *strain*, and (2) *a change of motion* termed *acceleration*. Hence the definition: Force is that which changes or tends to change the state of rest or motion of a body. The two kinds of effects just mentioned are perfectly distinct, and each furnishes us with means to measure

forces. Furthermore, whenever one body presses or pulls, attracts or repels, another, it is acted upon in an exactly similar manner by the second body with a force exactly equal to that which itself exerts, so that our conception of force should include the principle of duality or that forces always come into existence in pairs, each pair consisting of two equal and opposite forces, or that an action is always accompanied by an equal and contrary reaction. Action and reaction constitute what is termed *stress*, and according as these opposing forces act away from or towards each other, a stress is said to be a *tension* or *pressure.*

If a force f be applied to a body at rest for unit time we find that the body moves and has an acceleration g_1, and that if the same force is now made to act upon another body at rest for the same time and in consequence it takes an acceleration g_2, which is not equal to g_1, we express this fact by saying that the two bodies possess different quantities of matter or different masses; mass being defined as quantity of matter. There is thus an intimate connexion between acceleration and mass—and by experiment we find that the greater the mass the less the acceleration—corresponding to the application of a certain force. Mass may, therefore, be defined as a quantity inversely proportional to the acceleration which a given force f communicates to it, or $m = k \times \frac{1}{g}$, from which $mg = k$. And if we take as unit of mass that mass which when acted on by unit force takes unit acceleration, the constant k becomes unity; consequently $k = f$, and we have as a general law

$$f = m \,.\, g.$$

The fundamental law of gravitation states that masses attract each other, and when a body falls freely in vacuo with an acceleration g, the force of the earth's attraction or gravity on the mass is termed *weight.* The weight of a body is therefore proportional to its mass, and

$$\mathrm{w} = m \,.\, g$$

and a pounds-weight is a force, or the pull of the earth on the mass of a pound. Since masses, i.e. quantities of matter, are compared by comparing their weights much confusion exists, and the student is advised to distinguish between these two distinct physical conceptions, mass and weight. The pound and gramme are measures of mass, and the pounds-weight and grammes-weight are measures of force. Since g or the acceleration due to gravity is found by experiment to be 32·2 feet per second per second or 981 centimetres per second per second, the weight of a pound is

$$\begin{aligned} \mathrm{w}_p &= m \,.\, g = 1 \times 32{\cdot}2 \\ &= 32{\cdot}2 \text{ units of force.} \end{aligned}$$

The unit of force referred to here is the *poundal*, which is defined as that force which gives to a mass of one pound an acceleration of one foot per second per second.

And the weight of a gramme is

$$\begin{aligned} w_g &= m \cdot g = 1 \times 981 \\ &= 981 \text{ dynes.} \end{aligned}$$

When the weight of a body is taken as a unit of force it is termed a *gravitation* unit, but the objection to such a unit is that it has different values at different points of the earth's surface, and consequently the acceleration due to gravity, or g, varies in different localities, and it is thus necessary to state the place where the action takes place to evaluate the force. The weight of a gramme, for instance, is 981 dynes at Paris, 983 at the Poles, and 978 at the Equator.

Work. When a force is applied to a body and the point of application is displaced under the action of the force, work is said to be done. If the body moves in the *sense* in which the force acts, work is said to be done *by* the force, but if the displacement takes place in the *opposite sense*, work is done *against* the force. Work is measured by the product of the force and distance through which its point of application is moved, or

$$W = f \times s$$

where W = work, f = force, and s = distance moved through.

Then again when an agent overcomes resistance work is done, and the work done is measured by the product of the resistance, or opposing force, overcome into the distance through which it is overcome. A common example of work being done is when a body is raised vertically, in which case the weight of the body, as we have previously shown, is the force overcome. In all cases unit of work is done when unit force is overcome through unit distance; thus a *foot-pound* of work is done when a mass of one pound is raised vertically one foot. This is a gravitational unit much used by engineers, but it is not an absolute unit, since the weight of a pound, the force overcome, is 32·2 poundals, and to be exact it is necessary to state the place where the mass is raised before a precise measure can be obtained. In the C.G.S. system the unit of work is the work done by a dyne (unit force) in moving its point of application over a centimetre (unit distance), and is called an *erg*. The erg is obviously a very small unit, since the dyne is $\frac{1}{981}$ (or approximately $\frac{1}{1000}$) of the weight of a gramme, and the erg is slightly less than the work done in raising a cubic millimetre of water vertically one centimetre, and for practical purposes a secondary unit, equal to

ten million ergs, is taken as a unit of work and is called a *Joule.* Thus

$$1 \text{ joule} = 10{,}000{,}000 = 10^7 \text{ ergs.}$$

So far the time taken to do a unit of work has not entered into our considerations, but the rate at which an agent can do work is an important quantity and is termed *power.* *Power* is thus *rate of working*, and the unit of power is that of an agent which can do a unit of work in unit time. In the C.G.S. system of units the unit of power is the *erg per second*, but since the erg is a small unit of work a secondary unit of power, equal to the joule per second or ten million ergs per second, is mostly used in practice, and is termed the *Watt*, therefore

the watt is the rate of working equal to 10^7 ergs per second.

The unit of power in common use among engineers is the horse-power, which is the rate of working of an agent which performs 33,000 foot-pounds of work per minute or 550 foot-pounds per second. The connexion between the watt and horse-power has already been given as 746 watts are equal to one horse-power (see p. 54).

Energy. The general expression for work may be put in another form which will enable us to obtain a clear conception of the term energy. Thus we have shown that to give to a body of mass m at rest a velocity v at the end of one second, a certain force f must be applied to that body for unit time, and in that time the body is displaced a distance s, and the force f does a definite amount of work, equal to $f \times s$. Now at starting the velocity of the body was zero, and since it was v at the end of one second, the average velocity during the second would be $\frac{1}{2} v$. And the distance moved through is given by the product of the average velocity into the time, therefore

$$s = \frac{v}{2} \times 1 = \frac{v}{2} \text{ units of length.}$$

But the velocity acquired during that second is the same as the acceleration given to that body, and $g = v$, from which we get the measure of the force as

$$f = m \,.\, g = mv,$$

and the work done is

$$\begin{aligned} \text{W} &= f \,.\, s = (mv) \times \frac{v}{2} \\ &= \tfrac{1}{2} mv^2. \end{aligned}$$

This quantity representing the amount of work done in bringing the body to its actual state—i.e. moving with a velocity v—denotes the amount of work which the moving body can do in being stopped, and for this reason this quantity, $\frac{1}{2}mv^2$, is called *Kinetic Energy.* The *Energy* of a body is its *capacity* for doing work, and is

measured by the number of units of work it is capable of performing in passing from its existing state into some standard condition. When the energy possessed by a body is due to its motion it is termed *Kinetic Energy.* It thus follows that whenever a body enters into motion under the influence of a given force the work dispensed is equal to the increase of its kinetic energy.

Then again, when a body of mass m is raised vertically a height, h, against gravity the work done is

$$\begin{aligned} W &= f \times s = m g \times h \\ &= mgh \end{aligned}$$

and because work has been done on the body in raising it from the ground it has received an amount of energy equal to mgh, which is the amount of work it could do if allowed to fall. When raised the body possesses potentiality or energy in virtue of its new position relatively to the earth, and not on account of its velocity. Energy then may exist potentially, and the capability of a body to do work in virtue of its position or condition, and which may be dispensed at any time, is called *Potential Energy.* The work done by a projected cannon ball on a target is due to the kinetic energy of the moving ball, and the destructive effect is indicative of the energy of a moving body, but the work done by a bent spring is due to the stored up (potential) energy of the bent spring.

Principle of the Conservation of Energy. The algebraical equality between work and kinetic or potential energy deduced above introduces us to an important principle to which reference is frequently made in all branches of applied science. Instead of connecting work and energy by means of an equation, i.e. expressing the numerical values of two distinct physical quantities by an abstract mathematical equality—it would certainly be more in conformity with the idea of the transformation of one form of energy into another to state the connexion as a concrete physical equality, by saying that *work is transformed into kinetic or potential energy.* The property of transformation, in fact, is a characteristic principle of energy; and experiment clearly shows that the kinetic energy of body may be changed into potential energy, and vice versa, and also that the gain of one is equal to the loss of the other. An oscillating pendulum bob is an example of this principle. This idea implies that whilst energy may be transformed and readily change its form the amount of energy remains the same and is unalterable. A very special case occurs when energy is spent upon a body without communicating to it any velocity, but a thermal effect is produced and the temperature of the body is raised. Thus, in the case of a railway train moving with a certain velocity a large amount of kinetic energy is changed

into an equivalent amount of thermal energy or heat when the brakes are applied and the train is brought to rest. In all cases when work is done against opposing forces, friction for example, the amount of heat produced is always equivalent to the amount of work done. This is proved by experiment. Heat is in reality a form of kinetic energy due to the motion of molecules, and the change which results when kinetic energy is transformed into thermal energy is simply a change from a form of visible kinetic energy to a form of invisible kinetic energy. Sufficient has been said to indicate that we can distinguish quite a number of forms which energy may assume, and applied electricity deals very largely with the connexion existing between mechanical, thermal, chemical and electrical forms of energy; in fact, the part played by electricity in the arts is that of an agent transmitting and transforming different forms of energy.

The constancy of amount, which as we have shown is a characteristic property of energy, indicates that energy possesses the property of *Conservation*, and since conservation is the great test of reality, it assumes the part of a natural law of supreme importance, known as the *Principle of the Conservation of Energy*, which in Maxwell's words is as follows:

The total energy of a system is a quantity which can neither be increased nor diminished by any actions between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.

Heat is a form of energy, and it may be transformed into mechanical work and other forms of energy. Rumford, Joule and others have experimentally determined the amount of work which can be produced from a given amount of heat, and the amount of work which a unit of heat can perform dynamically is known as the *Mechanical Equivalent of Heat*, and is denoted by the letter J. Joule gave 772 for J, which means that the pound-degree-Fahrenheit heat unit can do 772 foot-pounds of work. More recent determinations give 780. The gramme-degree-Centigrade heat unit can do mechanically $4{\cdot}2 \times 10^7$ ergs of work, i. e. $J = 4{\cdot}2 \times 10^7$ in the C.G.S. system of units.

§ 73. **The C.G.S. or Absolute System of Units.** The C.G.S. absolute unit of force, or the *dyne*, was introduced by the British Association, and is a derived unit. It is defined by the relation $f = m \times a$ which connects the force f with the acceleration, a, impressed by it upon a mass m. Thus the dyne is that force which gives to a mass of one gramme an acceleration of one centimetre per second per second. The advantage possessed by this definition of rendering the unit of force independent of gravity, which is variable

according to the place where the action takes place, is that calculations are simplified and the unit is theoretically more perfect, and consequently the electric and magnetic units which are defined in terms of the dyne are rendered more exact and precise than if the unit of force were dependent upon gravity. The dyne is obviously a very small unit; in fact, gravity acting upon a mass of one gramme exerts a certain effort at Paris, which we may term the gramme-weight, and which produces an acceleration of 981 centimetres per second per second during its fall. Since the ratios of the accelerations which different forces impress upon the same mass is the same as the ratios of the forces, it follows from the relation $f = m \times a$, that the dyne, which impresses an acceleration of one centimetre per second per second, is $\frac{1}{981}$th of the force exercised by gravity, and

$$1 \text{ dyne} = \frac{1}{981} \text{ weight of a gramme.}$$

Thus, the dyne is dynamically approximately equal to the weight of a milligramme.

The C.G.S. unit of work is the *erg*, which is the work performed when a force of one dyne is overcome through a distance of one centimetre, so that an erg of work is done (approximately) when a milligramme is raised vertically a distance of one centimetre. And since power is the rate of working, the C.G.S. unit of power is the erg per second.

§ **74. Dimensions.** Absolute systems of units other than the C.G.S. system have been proposed and used, and in all cases the relations existing between the derived units and the fundamental units give us expressions which show algebraically the dimensions of such units with respect to the fundamental units of length, mass and time. The resulting equations showing the connexion between derived units and the fundamental units are termed dimensional equations; thus the following expressions or dimensional equations exhibit the dimensions and the manner in which the fundamental units enter into the derived units of the most important cases in Mechanics. The dimensions of the derived units may be defined as the powers to which the fundamental units of length, mass and time are raised in their dimensional equations.

Let [L], [M], [T] be the symbols denoting the three fundamental units, length, mass and time respectively, and also let v, a, M, f, W and P denote respectively, velocity, acceleration, momentum, force,

work and power, then since $Q = n[Q_0]$ according to first principles, we have for

Velocity $$v = n[v_0]$$

but velocity is space per time or $v = \frac{\text{space}}{\text{time}}$

$$\therefore \quad v = n\left[\frac{L}{T}\right] = n[LT^{-1}]$$

Acceleration $$a = n[a_0]$$

but acceleration is velocity per time or $a = \frac{\text{velocity}}{\text{time}} = \frac{\text{space}}{(\text{time})^2}$

$$\therefore \quad a = n\left[\frac{L}{T^2}\right] = n[LT^{-2}]$$

Momentum $$m = n[m_0]$$

but momentum is measured by the product of the mass into the velocity or $m = M \times v = \frac{\text{mass} \times \text{space}}{\text{time}}$

$$\therefore \quad m = n[LMT^{-1}]$$

Force $$f = n[f_0]$$

but force is measured by the product of the mass into the acceleration

$$\therefore \quad f = n[LMT^{-2}]$$

Work $$W = n[W_0]$$

but work is measured by the product of the force into the distance

$$\therefore \quad W = n[L^2MT^{-2}]$$

Power or *Activity* $$P = n[P_0]$$

but power is work per time or rate of working

$$\therefore \quad P = n[L^2MT^{-3}].$$

These examples indicate that the dimensions of different derived units bear certain ratios to each other, and that physical relationships are independent of the units chosen to represent them and are homogeneous with regard to the fundamental units. The following example illustrates how we may change from one system of units to another:

What will be the value of a force of 500 dynes in terms of units of a system of units in which the kilogramme is the unit of mass, the metre the unit of length, and the minute the unit of time?

Since $$f = n[LMT^{-2}]$$

$$f = 500\,[(\text{gramme})\,(\text{centimetre})\,(\text{second})^{-2}]$$

Similarly $f = n^1 [(\text{kilogramme})(\text{metre})(\text{minute})^{-2}]$

and $$n^1 = 500\left[\left(\frac{\text{gramme}}{\text{kilogramme}}\right)\left(\frac{\text{centimetre}}{\text{metre}}\right)\left(\frac{\text{minute}}{\text{second}}\right)^2\right]$$

$$= 500 \times \frac{1}{1000} \times \frac{1}{100} \times 60^2$$

$$= \frac{5 \times 36}{10} = 18.$$

Dimensional equations have several important functions: they

(1) facilitate the conversion of physical quantities from one system of units to another;

(2) enable us to verify the homogeneity of formulae;

(3) enable us to test equations of definitions, since by reducing both members to the fundamental units the equation should become an identity; and

(4) enable us to predict the form of a function when the physical quantities which enter into it are known.

The nature of a physical quantity is completely defined if we know its relation to its three fundamental units, and thus these formulae play in physical theories a part somewhat analogous to that played by chemical formulae in the science of chemistry. The latter show the relation of chemical compounds to certain fundamental kinds of matter. The former show the relation of complicated physical quantities to the three fundamental units. Very often the ratios of different dimensions have an obvious physical meaning, but this is not always the case; thus the ratio of volume to area is length, or

$$\frac{\text{volume}}{\text{area}} = \frac{[L^3]}{[L^2]} = [L] = \text{length}$$

but the ratio of a force to a velocity has no obvious physical meaning, since

$$\frac{\text{force}}{\text{velocity}} = \frac{[LMT^{-2}]}{[LT^{-1}]} = [MT^{-1}].$$

Then again dimensional formulae as explained so far fail in one respect, i.e. in that they are not capable of expressing direction. As Professor S. P. Thompson has pointed out the dimensions for work and moment of a force are the same, so that the same dimensional formula may express two quite different quantities. Both work and moment of a force about a point are of the nature of a force multiplied by a distance, and hence the dimensions of both of them are $[L^2MT^{-2}]$. This difficulty makes it necessary to extend the ordinary dimensional formulae, so as to make them include the idea of direction. In the case of work the force and the distance are

measured in the same direction; in the case of a moment their directions are mutually perpendicular. Mr. Williams has shown that the idea of direction may be introduced by selecting any three arbitrary directions which are mutually at right angles, and indicating the unit of length measured along these three directions by X, Y, Z respectively. X, Y, Z are therefore three equal lengths, and the difference of symbol has reference only to the difference of direction. If we suppose the force to act in the direction X, then

the dimensional expression for work is $[X^2MT^{-2}]$

and the dimensional expression for moment is $[XYMT^{-2}]$ or $[XZMT^{-2}]$

according as the perpendicular of the force is drawn parallel to the direction of Y or Z.

§ 75. **Electrical and Magnetic Units.** An absolute system of electrical and magnetic units may be deduced from any of the equations expressing the fundamental laws of electricity and magnetism, and the corresponding dimensional expressions depend upon the particular relationship taken as a starting point. As Maxwell pointed out in 1863 the electrical phenomena susceptible of measurement are four in number, i.e. quantity, current, electromotive-force and resistance, and very simple relations exist between them. In practice two absolute systems of electrical and magnetic units have been built up, i.e. (1) the Electrostatic System (E.S.U.), which depends upon the definition of unit quantity of electricity, and (2) the Electromagnetic System (E.M.U.), which depends upon the definition of unit magnetic pole. Thus, by taking as a starting point the experimental results of Coulomb, which connect both electrical quantities and magnetic poles with dynamical force, we may obtain expressions for current, resistance, electromotive force, &c., by the reductions to dynamical units of (1) electrostatic attractions and repulsions which give the Electrostatic absolute system of units, and of (2) electromagnetic actions produced in galvanic circuits which give the Electromagnetic absolute system of units.

Taking the Electrostatic System first, the experimental result obtained by means of Coulomb's torsional balance, i. e. that the force exerted between two small electrified bodies is directly proportional to the product of their charges and inversely proportional to the square of the distance between them, gives the relation

$$f \propto \frac{q \cdot q^1}{d^2} \quad \text{or} \quad f = \text{constant} \times \frac{qq^1}{d^2}\cdot$$

The introduction of a constant is necessary, for, as Faraday proved, electrical influences are transferred more readily by some media than by others, and the magnitude of the force exerted by the two

charges depends upon the transmitting power of the medium, or the specific inductive capacity of the medium as it is termed, as well as upon the inverse square of the distance between them. If K denotes the specific inductive capacity of the medium, then the constant is $\frac{1}{K}$. For purposes of comparison air is taken as the standard medium, so that if unit quantity of electricity be concentrated at a point one centimetre from an equal quantity of electricity in air the force exerted between them is one dyne, since K is unity.

If the two charges are equal the general expression is

$$f = \frac{1}{K}\frac{q^2}{d^2}$$

from which $$q = \sqrt{fKd^2}$$

and the dimensional equation is $[q] = [LMT^{-2} . K . L^2]^{\frac{1}{2}}$

$$= [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}].$$

In a similar manner the dimensional equations for the other quantities are obtained as follows :—

Electric Surface Density, σ, denotes the degree of electrification at any point of a conductor, and is measured by the quantity of electricity per unit area, therefore

$$\sigma = \frac{q}{A}$$

and the dimensional formula is $[\sigma] = [q . L^{-2}]$

$$= [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}} \times L^{-2}]$$
$$= [L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}].$$

Intensity of Electric Field, f_1, is the force with which a unit of positive electricity would be acted on if placed at the point under consideration, therefore

$$f = f_1 q \quad \text{and} \quad f_1 = \frac{f}{q}$$

and the dimensional equation is

$$[f_1] = [f . q^{-1}] = [LMT^{-2} \times L^{-\frac{3}{2}}M^{-\frac{1}{2}}TK^{-\frac{1}{2}}]$$
$$= [L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}].$$

Difference of Electric Potential, V, is measured by the work done if a unit of positive electricity were made to pass from a point of lower potential to one of higher potential against the electric forces, therefore,

work done on q units $= W = qV$

and $$V = \frac{W}{q}$$

and the dimensional expression is

$$[V] = [Wq^{-1}] = [L^2MT^{-2} \times L^{-\frac{3}{2}}M^{-\frac{1}{2}}TK^{-\frac{1}{2}}]$$
$$= [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}].$$

The Capacity of a Conductor, k, is the quantity of electricity required to charge a conductor to unit potential, therefore

$$q = kV \quad \text{and} \quad k = \frac{q}{V}$$

and the dimensional expression is

$$[k] = [q \,.\, V^{-1}] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}} \times L^{-\frac{1}{2}}M^{-\frac{1}{2}}TK^{\frac{1}{2}}]$$
$$= [LK].$$

The Electric Current, C, is that quantity of electricity which passes across a given cross-section per unit of time, therefore

$$q = Ct \quad \text{and} \quad C = \frac{q}{t}$$

and the dimensional equation is

$$[C] = [qt^{-1}] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}} \times T^{-1}]$$
$$= [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}].$$

The Electrical Resistance, R, of a conductor is the ratio of the difference of potential at its ends and the current traversing it, therefore

$$R = \frac{V}{C}$$

and the dimensional formula is

$$[R] = [VC^{-1}] = [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}} \times L^{-\frac{3}{2}}M^{-\frac{1}{2}}T^2K^{-\frac{1}{2}}]$$
$$= [L^{-1}TK^{-1}].$$

Electrical Conductance is the reciprocal of resistance and its dimensional expression is $[LT^{-1}K]$.

In a similar manner all magnetic and electromagnetic quantities are expressed in units which are derived from the definition of unit magnetic pole. In other words the fundamental entity of the electromagnetic system is a quantity of magnetism imagined to be concentrated on an ideal point-pole and producing a mechanical attraction or repulsion upon a similar point-pole at a distance, and upon which the system has been built up as a basis. The variable transmitting power of different media is also recognized, and the magnetic inductive capacity or the *magnetic permeability* of the medium, as it is called, is denoted by the symbol μ. Therefore the force exerted before two magnetic poles m and m_1, d centimetres apart, is given by

$$f = \frac{1}{\mu} \cdot \frac{m\,m_1}{d^2}$$

and since unit magnetic pole depends upon the C.G.S. unit of force, unit magnetic pole is defined as that pole which exerts a force of one dyne upon an equal pole one centimetre distant placed in air. The general expression is therefore

$$f = \frac{1}{\mu}\frac{m^2}{d^2}$$

and

$$m = \sqrt{f\mu d^2}$$

and the dimensional expression is

$$[m] = [f^{\frac{1}{2}} \cdot \mu^{\frac{1}{2}} d] = [\mathrm{L}^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1} \times \mu^{\frac{1}{2}} \times \mathrm{L}]$$
$$= [\mathrm{L}^{\frac{3}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{\frac{1}{2}}].$$

Intensity of Magnetic Field, H, due to sources of magnetism, is at every point measured by the force which a unit magnetic pole would experience if placed at that point, therefore

$$f = \mathsf{H}m \quad \text{and} \quad \mathsf{H} = \frac{f}{m}$$

and the dimensional formula is

$$[\mathsf{H}] = [fm^{-1}] = [\mathrm{LMT}^{-2} \times \mathrm{L}^{-\frac{3}{2}}\mathrm{M}^{-\frac{1}{2}}\mathrm{T}\mu^{-\frac{1}{2}}]$$
$$= [\mathrm{L}^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{-\frac{1}{2}}].$$

The Moment of a Magnet, M, is given by the product of the pole strength into the distance between the poles, therefore

$$\mathsf{M} = lm$$

and the dimensional equation is

$$[\mathsf{M}] = [l \,.\, m] = [\mathrm{L} \times \mathrm{L}^{\frac{3}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{\frac{1}{2}}]$$
$$= [\mathrm{L}^{\frac{5}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{\frac{1}{2}}].$$

The Intensity of Magnetization, I, is the magnetic moment of a magnet per unit volume, therefore

$$\mathsf{I} = \frac{\mathsf{M}}{\mathrm{v}}$$

and the dimensional expression is

$$[\mathsf{I}] = [\mathsf{M}\mathrm{v}^{-1}] = [\mathrm{L}^{\frac{5}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{\frac{1}{2}} \times \mathrm{L}^{-3}]$$
$$= [\mathrm{L}^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-1}\mu^{\frac{1}{2}}].$$

To obtain the connexion between the electrical and magnetic quantities advantage is taken of the experimental facts, proved by Oersted and Ampere, that a current of electricity traversing a linear conductor radiates magnetic wave systems and sets up a magnetic field around the conductor. If the conductor be formed into a circle of radius r centimetres then a magnetic pole placed at its centre experiences a definite force which experiment proves to be

$$f = \frac{2\pi r\mathrm{C}}{r^2} = \frac{2\pi\mathrm{C}}{r},$$

where C is the current. It follows from this, by definition, that the

C.G.S. absolute unit of current is such that it exerts a force of one dyne on a unit magnetic pole placed at the centre of a circle of one centimetre radius, traversed by it, per unit length of the current. In other words, unit current is such that it exerts a total force of one dyne on a unit magnetic pole placed at the centre of a circle of 2π centimetres radius traversed by the current. To obtain the dimensional expression of an electric current in the electromagnetic system of units we have only to remember that the intensity of a magnetic field is measured by the force which will be experienced at the point under consideration to get the following relation :—

$$\mathsf{H} = \frac{f}{m} = \frac{2\pi C}{r}$$

from which

$$C = \frac{1}{2\pi} \cdot \frac{fr}{m}$$

and the dimensional expression is

$$[C] = [f \,.\, r \,.\, m^{-1}] = [LMT^{-2} \times L \times L^{-\frac{3}{2}}M^{-\frac{1}{2}}T\mu^{-\frac{1}{2}}]$$
$$= [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}].$$

The Quantity of Electricity, Q, is given by the product of the current into the time during which it lasts, therefore

$$Q = C \,.\, t$$

and the dimensional equation is

$$[Q] = [C \,.\, t] = [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}} \times T]$$
$$= [L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}].$$

The Electromotive-force, E, is measured by the work done in transferring unit quantity of electricity from one point to another, therefore

$$W = QE \quad \text{and} \quad E = \frac{W}{Q}$$

and the dimensional formula is

$$[E] = [WQ^{-1}] = [L^2MT^{-2} \times L^{-\frac{1}{2}}M^{-\frac{1}{2}}\mu^{\frac{1}{2}}]$$
$$= [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}].$$

Electrical Resistance, R, is given by the ratio of E to C, therefore

$$R = \frac{E}{C}$$

and the dimensional expression is

$$[R] = [EC^{-1}] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}} \times L^{-\frac{1}{2}}M^{-\frac{1}{2}}T\mu^{\frac{1}{2}}]$$
$$= [LT^{-1}\mu].$$

The Capacity of a Conductor, k, is the quantity of electricity required to charge the conductor to unit potential, therefore

$$Q = k \,.\, E \quad \text{and} \quad k = \frac{Q}{E}$$

and the dimensional expression is

$$[k] = [Q \,.\, E^{-1}] = [L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}} \times L^{-\frac{3}{2}}M^{-\frac{1}{2}}T^2\mu^{-\frac{1}{2}}]$$
$$= [L^{-1}T^2\mu^{-1}].$$

It will have been observed that the electromagnetic system of units does not coincide with the electrostatic system, and the difference between their respective dimensional expressions is indicated in the following table, in which K and μ are removed.

Unit.	Dimensions in E.S.U.	Dimensions in E.M.U.	Ratio of E.S.U. to E.M.U.
Quantity	$L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}$	$L^{\frac{1}{2}} M^{\frac{1}{2}}$	$L\,T^{-1} = v$
Current	$L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}$	$L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}$	$L\,T^{-1} = v$
Difference of Potential	$L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}$	$L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}$	$L^{-1}\,T = \frac{1}{v}$
Resistance	$L^{-1}\,T$	$L\,T^{-1}$	$L^{-2}\,T^2 = \frac{1}{v^2}$
Capacity	L	$L^{-1}\,T^2$	$L^2\,T^{-2} = v^2$

One electromagnetic unit of quantity . . . $= v$ electrostatic units
" " " " current . . . . . $= v$ " "
" " " " difference of potential $= \frac{1}{v}$ " "
" " " " resistance . . . $= \frac{1}{v^2}$ " "
" " " " capacity . . . . . $= v^2$ " "

As shown in the fourth column the ratio of the dimensions of a unit in the electrostatic system to its dimensions in the electromagnetic system is expressed as a definite velocity, and numerous experiments give the numerical value of v approximately 3×10^{10} centimetres per second. It is to be observed that this value of v is the same as the value of the velocity of propagation of light *in vacuo*, and on Maxwell's theory electric disturbances are propagated with the velocity of light.

§ 76. **Practical Electrical Units.** In the absolute C.G.S. system of units the unit of E.M.F. does one erg of work per second in sending the C.G.S. unit of current through a circuit, and the C.G.S. unit of resistance is such that one erg of work is expended per second when a C.G.S. unit of current traverses it under a pressure of one C.G.S. unit of pressure. The magnitude of the erg is very small, consequently the C.G.S. units of electrical pressure and resistance are far too small for practical purposes, nor could any standard of the

C.G.S. unit of resistance conveniently be determined with accuracy, and for these reasons the Paris Congress of 1881 decided upon the practical system of units with the volt, ampere and ohm as the units of E.M.F., current and resistance respectively, and which has since been legalized. The volt was fixed as 100 million times the C.G.S. of pressure, probably because of its close approximation to the E.M.F. of a Daniell's cell, which at that time was considered as a representative standard of pressure. Similarly, the ohm was specified to be 1000 million times as large as the C.G.S. unit of resistance, probably because of its close approximation to the value of the Siemens mercury unit and standard of resistance. By Ohm's Law the magnitude of the practical unit of current, the ampere, followed by direct sequence to be $\frac{1}{10}$th the C.G.S. unit of current.

The effect of selecting the above arbitrary multipliers of the C.G.S. units for the practical units, the volt and ohm, while retaining the C.G.S. unit of time (the second), was equivalent to replacing the centimetre by a unit of length equal to an arc of the earth's surface of 1000 million centimetres, and replacing the gramme by a mass one-billionth of a centigramme or 10^{-11} gramme. For this reason the practical system of electrical units is sometimes termed the quadrant-eleventh-et-gramme-second system. Then again the experience of a quarter of a century and the development of electrical work have made it apparent that the present system is defective; it is obvious that for modern requirements the most suitable unit for magnetic-pole would be one which produces and emits unit magnetic flux, instead of the unit pole acting on its prototype with the force of one dyne when placed one centimetre away. It would also conform with modern theory in explaining magnetic phenomena in space by disturbances of the medium rather than by some action at a distance. Similarly, a unit electric flux would be a more natural starting point than the present unit electric charge for the electrical units. Obviously, considerable simplification would follow if the number of the lines of force emitted from the pole of a magnet were m instead of $4\pi m$ as at present, and the practical man certainly prefers 'ampere-turns' to '$\cdot 4\pi \times$ ampere-turns' as the excitation of the coil of an electromagnet, and in fact very often adopts it. It thus appears that the British Association committed a grave error of judgement when defining the unit magnetic pole and unit quantity of electricity on the assumption that the resulting actions take place between pole-points and charged-points, and thus ignored the properties and existence of the medium, and neglected to take into account the effects of permeability and specific inductive capacity. The effect is that electrical equations are complicated and disfigured by the

4π-eruption, as it is familiarly called, and that there are two different systems of units, the E.S.U. and E.M.U., which do not coincide in the least, but which are such that in one a unit is thirty thousand million centimetres per second greater than in the other. Rücker has shown that although the dimensions of K and μ, the dielectric constant and permeability respectively, are unknown there is a relation between electricity and magnetism which indicates that these quantities are related to each other. The force exerted between a current and a magnetic pole points to this fact, and by assuming that $K \times \mu = v^2$, the dimensional expressions would be rationalized by inserting these constants in them.

§ 77. **Rational Systems.** The want of uniformity in the two systems of electrical units has given rise to much discussion and several proposals have been made to make some radical changes in these systems so as to bring them into harmony with present requirements. Much may be said for and against making these changes, and the following is a brief outline of the most important proposals. The first to expose the imperfections of the British Association system and at the same time to suggest a remedy was Mr. Oliver Heaviside, who so long ago as 1882 expounded his system of rational units. His starting point was a new definition of unit magnetic pole, in which unit magnetic pole was defined to be one which exerts a force of 4π dynes on an equal pole one centimetre distant. The essential point in his system is that magnetic and electric phenomena are manifestations of stresses and strains in the medium, and as Professor Fleming has stated in the *Electrician*, 'Mr. Heaviside, starting from his new definition, evolves a series of electrical and magnetic units, different from the British Association system units in size, but characterized by the property that there are no unmeaning constants in the equations expressing their relations.' The relations of the Heaviside units to the British Association units in magnitude are as follows:

One Heaviside unit of

Magnetic pole . . $= \dfrac{1}{\sqrt{4\pi}} \times$ C.G.S. unit magnetic pole,

Magnetic force . . $= \sqrt{4\pi} \times$ C.G.S. unit of magnetic force,

Magnetic flux . . $= \sqrt{4\pi} \times$ C.G.S. unit of flux (= 1 line),

Current . . . . . $= \dfrac{10}{\sqrt{4\pi}} \times$ the present ampere,

Electromotive-force $= \sqrt{4\pi} \times$ the present volt,

One Heaviside unit of

Resistance . . . $= \frac{4\pi}{10} \times$ the present ohm,

Inductance . . . $= \frac{4\pi}{10} \times$ the present henry,

Quantity . . . . $= \frac{10}{\sqrt{4\pi}} \times$ the present coulomb,

Capacity . . . . $= \frac{10}{4\pi} \times$ the present farad,

Magnetomotive-force $= \frac{10}{\sqrt{4\pi}} \times$ the present ampere-turn,

Power . . . . . . $= 10 \times$ the present watt,

Energy . . . . . $= 10 \times$ the present joule.

The perfection of Heaviside's system is acknowledged on all sides, but its introduction would involve so radical a change in existing standards, apparatus, agreements and literature, that few are ready to advocate the change.

Professor Fessenden has suggested an alternative proposal, and suggests that the formulae be rationalized instead of the units. This could be accomplished by assuming that the permeability of air shall be numerically 4π instead of unity, and since the permeability of air is not an absolute value the change would only affect the tabulated values of the permeability of iron and steel, whilst the 4π would be removed from our formulae. The ampere, coulomb, volt, ohm, farad, henry, joule, and watt would retain their present values, whilst

the unit of magnetism	would become	$\frac{1}{4\pi}$	its present value.
,, ,, ,, permeability	,, ,,	4π	
,, ,, ,, specific inductive capacity	,, ,,	$\frac{1}{4\pi}$	
,, ,, ,, magnetic potential . .	,, ,,	4π	its present value.

These changes would certainly tend to simplicity, but up to the present no official endorsement has been made, and although an opportunity presented itself at the International Congress at Paris, held in 1900, we are still troubled with the irrepressible 4π in our formulae and a most irrational system of units.

Worked Examples. (1) If 10 feet per second be taken as the unit of velocity, and the acceleration due to gravity be 50 feet per second per second the unit of acceleration, find the magnitude of the units of length and time.

Since the dimensions for velocity and acceleration are

$$[v] = [\mathrm{L\,T}^{-1}] \quad \text{and} \quad [a] = [\mathrm{L\,T}^{-2}]$$

we have $$[10] = [l\,t^{-1}] \quad \text{and} \quad [50] = [l\,t^{-2}]$$

$$\therefore \quad \frac{50}{10} = \frac{l\,t^{-2}}{l\,t^{-1}} = \frac{1}{t}$$

and $$t = \tfrac{1}{5} \text{ second (unit of time).}$$

Substituting this value we have

$$10 = \frac{l}{\frac{1}{5}} = 5\,l$$

$$\therefore \quad l = \frac{10}{5} = 2 \text{ feet (unit of length).}$$

(2) If the centimetre be taken as the unit of length determine the magnitude of the unit of time so that g may be unity.

Since $$[a] = [\mathrm{L\,T}^{-2}]$$

we have $$\text{acceleration} = 981\,[(\text{centimetre}) \times (\text{second})^{-2}]$$
$$= 1\,[(\text{centimetre}) \times t^{-2}$$

if $$= \text{unit of time}$$

then $$\frac{981}{1}\left[\frac{\text{cm.}}{\text{cm.}} \times \frac{t^2}{\text{second}^2}\right] = 1$$

$$\therefore \quad 981\,t^2 = 1$$

and $$t = \frac{1}{\sqrt{981}} = 0{\cdot}032 \text{ second.}$$

(3) A tramcar of weight 10 tons is running at the rate of 10 miles per hour, and it is found that it takes 5·2 seconds for a brake to bring it to rest after the current is cut off. Determine the force exerted assuming it to be uniform, the energy dissipated, and the distance travelled in pulling up.

$$\text{Velocity of 10 miles per hour} = \frac{10 \times 1760 \times 3}{60 \times 60}$$

$$= \frac{44}{3} \text{ feet per second.}$$

Then if a = the retardation, the velocity is equal to the retardation multiplied by the time or

$$v = a\,t \quad \text{and} \quad a = \frac{v}{t}$$

$$\therefore \quad a = \frac{\frac{44}{3}}{5{\cdot}2} = \frac{44}{3 \times 5{\cdot}2}$$

$$= \frac{11}{3{\cdot}9} \text{ feet per second per second.}$$

But $$\text{force} = m \times a,$$

and the mass of the tramcar is $10 \times 2240 = 22400$ pounds

$$\therefore \quad \text{force} = 22400 \times \frac{11}{3.9} \text{ poundals}$$

$$= 63179{\cdot}5 \text{ poundals.}$$

$$\text{The energy dissipated} = \tfrac{1}{2}\, m.\, v^2$$

$$= \tfrac{1}{2} \times 10 \times 2240 \times \left(\frac{44}{3}\right)^2$$

$$= 2409277{\cdot}7 \text{ foot-poundals}$$

$$= \frac{2409277{\cdot}7}{32{\cdot}2} \text{ foot-pounds}$$

$$= 74822{\cdot}3 \text{ foot-pounds}$$

$$\text{work} = \text{force} \times \text{distance}$$

$$2409277{\cdot}7 = 63179{\cdot}5 \times s$$

$$\therefore \quad s = 38\ 1 \text{ feet.}$$

(4) A motor develops 12·5 horse-power when driving a car at the rate of 10 miles per hour. Determine the resistance offered to motion.

$$\text{Velocity of 10 miles per hour} = \frac{44}{3} \text{ feet per second}$$

and 12·5 H.P. is equivalent to $12{\cdot}5 \times 550$ foot-pounds per second.

Let f_t be the required resistance (in pounds-weight) to motion, then the work done per second is

$$f_t \times \frac{44}{3} = 12{\cdot}5 \times 550 \text{ foot-pounds}$$

and
$$f_t = \frac{12{\cdot}5 \times 550 \times 3}{44} = 468{\cdot}75 \text{ pounds-weight.}$$

(5) If the magnitude of the unit of length were increased threefold and that of the unit of time were doubled, how would the C.G.S. electromagnetic unit of electric pressure be changed in magnitude?

Since the dimensional formula for electric pressure is

$$[\mathrm{E}] = [\mathrm{L}^{\frac{3}{2}}\, \mathrm{M}^{\frac{1}{2}}\, \mathrm{T}^{-2}]$$

we have, if n = the number of new units equal to one of the old units

$$1\,[\mathrm{L}^{\frac{3}{2}}\, \mathrm{M}^{\frac{1}{2}}\, \mathrm{T}^{-2}] = n\,[\mathrm{L}_1{}^{\frac{3}{2}}\, \mathrm{M}^{\frac{1}{2}}\, \mathrm{T}_1{}^{-2}]$$

$$= n\,[(3\,\mathrm{L})^{\frac{3}{2}}\, \mathrm{M}^{\frac{1}{2}}\, (2\,\mathrm{T})^{-2}]$$

$$= n\,[1.3\, \mathrm{L}^{\frac{3}{2}}\, \mathrm{M}^{\frac{1}{2}}\, \mathrm{T}^{-2}]$$

and
$$n = \frac{1}{1{\cdot}3} = 0{\cdot}77$$

from which it is clear that the magnitude of the new unit is 1·3 times that of the old, and the new voltage would be only 77 per cent. that of the old.

(6) Experiment has shown that the velocity of propagation of an undulatory movement in a medium depends on the modulus of elasticity and the density of the medium. Determine the relation existing between the velocity and the elasticity and density.

This example is taken from Gerard's 'Leçons sur l'Electricitie,' and illustrates how dimensional equations may assist us to arrive at the form of a function when the experimental investigation has been made. Since the

velocity of propagation is a function of the elasticity ϵ and density δ of the medium, we have

$$v = \phi\,(\epsilon \cdot \delta)$$

and by introducing the dimensions of the quantities named we have

$$v \,.\, \mathrm{L\,T^{-1}} = \phi\,[\epsilon \,.\, \mathrm{L^{-1}\,M T^{-2}},\ \delta\,\mathrm{L^{-3}\,M}].$$

Since M is absent from the left-hand side of this equation whilst it occurs in both the right-hand members and is raised to the same power, we are led to infer that the velocity is a function of the quotient of the elasticity by the density, and

$$v \,.\, \mathrm{LT^{-1}} = \phi\,\left\{\frac{\epsilon \,.\, \mathrm{L^{-1}\ MT^{-2}}}{\delta\,\mathrm{L^{-3}\ M}}\right\}$$

$$= \phi\,\left\{\frac{\epsilon}{\delta}\,\mathrm{L^2\,T^{-2}}\right\} = \phi\,\left\{\frac{\epsilon}{\delta}[\mathrm{L\,T^{-1}}]^2\right\}$$

This result indicates that the right-hand side of the equation will contain L and T raised to the same power as those of the left-hand side if we take the square root of the elasticity and density. In other words we may conclude that the velocity is a linear function of $\sqrt{\frac{\epsilon}{\delta}}$, or $v = \sqrt{\frac{\epsilon}{\delta}}$.

EXERCISES XI.

(1) Prove that 1 H.P. = 74,600 kilogramme-metres per second.

(2) How many joules of work are done on a mass of 0·75 cwt. when raised vertically through 150 feet?

(3) 50×10^8 ergs of work were expended in raising 100 kilogrammes; to what height was the body raised?

(4) Express a velocity of 40 miles per hour in centimetres per second.

(5) Express an acceleration of 40 feet per second per second in metres per minute per minute.

(6) 5×10^8 ergs of work are required to move a body 15×10^4 centimetres, what was the average force required?

(7) A body weighing 12 pounds is moving with a velocity of 193·2 feet per second. What constant force in dynes must be applied to bring it to rest in 3 seconds?

(8) The weight of 12 kilogrammes is overcome through a distance of 20 metres, determine the number of foot-pounds of work done.

(9) What will be the value of 500 dynes in terms of the units of a system of units in which the kilogramme is the unit of mass, the metre the unit of length, and the minute the unit of time.

(10) In two different systems of units an acceleration is represented by the same number, whilst a velocity is represented by numbers in the ratio 3 : 1. Compare the units of length and time, and also the numbers representing a force in the two systems.

(11) Has the following expression any meaning?

$$\text{work} = \frac{3\,m\,.\,v}{t} - 2\,\mathrm{F}\,vt.$$

(12) A certain current has a strength of 20 units in the electromagnetic system of units, what is its measure in a millimetre-milligramme-second system?

(13) What change will be made in the magnitude of the units of current, E.M.F. and resistance in the electromagnetic system if 10^7 metres be taken as the unit of length, $\frac{1}{10}$th gramme as the unit of mass and the second as the unit of time?

(14) The measure of a P.D. in the electromagnetic system of units is 10 volts, what will be the measure of the same P.D. in a foot-pound-minute system of units?

(15) In the E.M.U. system $\mathrm{H} = 0{\cdot}18$. What is this in the foot-grain-second system?

1 pound = 7000 grains; 1 gramme = 15·432 grains.

(16) The magnetic intensity of a field is 10·264 in the foot-grain-second system; what is its equivalent in the E.M.U. system?

(17) The magnetic moment of a magnet in the C.G.S. system of units is 1600. What is this in the foot-pound-second system?

(18) Find the number of volts in an electrostatic unit of potential.

(19) A runaway car, weighing 10 tons, is moving at the rate of 20 miles per hour, and is pulled up in a distance of 20 yards upon application of the emergency brake. Determine the energy dissipated, the average braking power, the retardation and the time taken in pulling up.

APPENDIX

USEFUL TABLES AND CONSTANTS

TABLE A.

PHYSICAL CONSTANTS.

Table of Specific Gravity, Weight, Melting Point, etc., of Metals.

Metal.	Specific Gravity.	Weight per Cubic Foot.	Specific Heat.	Melting Point in Degrees Fahrenheit.	Relative Conductivity.
Aluminium, Cast	2·5	156·06	·2143	...	...
" Hammered	2·67	166·67	...	...	54·2
Antimony	6·702	418·37	·0508	810·	3·88
Arsenic	5·763	359·76	·0814	365·	...
Barium	4·	249·70	...	...	...
Bismuth	9·822	613·14	·0308	497·	1·2
Brass, 35 per ct. Zinc	8·218	513·6	...	...	21·15
Cadmium	8·604	537·10	·0567	500·	...
Calcium	1·565	97·76	...	...	...
Chromium	7·3	455·70	...	...	...
Cobalt	8·6	536·86	·1070	...	...
Copper	8·895	555·27	·0933	1996·	100·
" Rolled	8·878	554·21	...	...	...
" Cast	8·788	548·59	...	...	...
" Drawn	8·946	558·47	...	...	...
" Hammered	8·958	559·25	...	...	...
" Pressed	8·931	557·52	...	...	...
" Electrolytic	8·914	556·46	...	...	...
Gold	19·258	1202·18	·0324	2016·	78·
Iron, Bar	7·483	467·18	·130	2786·	...
" Wrought	7·79	486·29	·113	3286·	16·
Steel	7·85	490·03	·116	3286·	...
Lead	11·445	714·45	·033	612·	8·88
Magnesium	2·24	139·83	·2499	...	...
Manganese	6·9	430·73	·114	3000·	...
Mercury	13·568	846·98	·0319	—38·	1·6
Nickel	7·832	488·91	·1091	2800·	7·89
Platinum	20·3	1267·22	·0324	3286·	10·6
Potassium	·865	54·00	·1696	136·	...
Silver	10·522	656·84	·0570	1873·	100·
Sodium	·972	60·68	·2934	194·	...
Strontium	2·504	156·31	...	...	...
Tin	7·291	455·14	·0562	442·	15·45
Zinc	6·861	428·29	·0955	773·	29·9

TABLE B.

SPECIFIC RESISTANCES.

Metals.	Microhms at 0° C. Per Inch Cube.	Microhms at 0° C. Per Cm. Cube.	Ohms per Mil-Foot.	Temperature Coefficient a $R_t = R_0(1 + a\,t)$
Silver, Annealed	0·578	1·468	8·80	0·004
,, Hard-drawn	0·634	1·620	9·76	0·004
Copper, Annealed	0·614	1·561	9·37	0·00428
,, Hard-drawn	0·634	1·621	9·76	0·004
Gold, Hard-drawn	0·865	2·197	13·2	0·00377
Aluminium, Annealed	1·49	2·665	16·0	0·00435
Zinc	2·26	5·751	34·6	0·00406
Platinum, Annealed	4·3	10·917	65·6	0·003699
Iron, Annealed	3·57	9·065	54·4	0·00625
Nickel	4·85	12·323	74·0	0·00622
Tin	5·13	13·048	78·5	0·0044
Lead	8·0	20·38	122·5	0·00411
Mercury	37·0	94·07	565·0	0·00072
Non-Metal.				
Carbon (arc lamp)	...	4400 to 8600	...	0·0005
Alloys.				
Brass	2·83	7·2	43·5	...
German Silver (4 Cu + 2 Ni + 1 Zn)	8·28	21·0	126·25	0·000273
Manganin (84 Cu + 12 Mn + 4 Ni)	16·56	42·0	252·5	0·000015
Platinoid (German Silver + 1 or 2 % Tungsten)	16·4	41·7	251·0	0·0003
Platinum Silver (1 Pt + 2 Ag)	12·4	31·582	190·0	0·0002
Manganese Steel	26·7	68·0	409·0	0·00122
Reostene	30·0	76·468	459·0	0·0011

TABLE C.

REPORT OF THE COMMITTEE ON COPPER CONDUCTORS.

The advantages of standardizing are being largely advocated for almost all classes of engineering products, but copper conductors have hitherto not been included in the list as every one has assumed that they were already standardized and that Matthiessen had settled the resistance and temperature coefficient of copper in his researches nearly forty years ago.

This assumption is far from being true and the catalogues of various electrical cable makers as published before 1899 show considerable discrepancies in the resistance and weight of nominally the same cables, while the Post Office issued a specification differing from all the others.

In order to remedy this confusion a committee was formed of those interested to determine a standard, and the Institution of Electrical Engineers, the General Post Office, and the principal manufacturers of rubber insulated cables, sent representatives. Meetings were held early in 1899, and the report given below was adopted by the following delegates:

Sir W. H. Preece, K.C.B., F.R.S., Chairman; Mr. J. Gavey, Mr. H. Hartnell (for the General Post Office); Prof. W. E. Ayrton, F.R.S., Mr. W. M. Mordey, Mr. Herbert A. Taylor (for the Institution of Electrical Engineers); Mr. J. W. Conolly (Conolly Bros., Limited); Mr. R. J. Hatton (Henley's Telegraph Works, Limited); Mr. W. E. Gray (I.R.G.P. and Telegraph Works, Limited); Mr. A. Paterson (Johnson & Phillips); Mr. F. Jacob, Mr. J. S. Huddlestone (Siemens Brothers, Limited); Mr. A. H. Howard (W. T. Glover & Co., Limited).

The Telegraph Manufacturing Company, Limited, and the London Electric Wire Company, Limited, also co-operated in forming the committee and have adopted its recommendations.

The report of the committee is as follows:

1. Resolved, That Matthiessen's standard of 0·153851 standard ohm resistance for a wire one metre long, weighing one gramme at 60° F., be taken as the standard for hard drawn high conductivity commercial copper.

2. Hard drawn copper to be defined as that which will not elongate more than 1 per cent. without fracture.

3. Resolved, That Matthiessen's standard of 0·150822 standard ohm resistance for a wire one metre long, weighing one gramme at 60° F., be taken as the standard for annealed high conductivity commercial copper.

4. Copper to be taken as weighing 555 lb. per cubic foot at 60° F. which will give a specific gravity of 8·912.

5. Resolved, That Messrs. Clarke, Forde, and Taylor's temperature coefficient, as published in their pamphlet, dated February 20, 1899, be adopted, and that the average coefficient of 0·00238 per degree F. be adopted for commercial purposes.

6. Resolved, That the resistance and weight of conductors be calculated from the actual length of the wires.

7. Resolved, That a lay of twenty times the pitch diameter be taken as the standard for the calculation of tables.

8. Resolved, That 2 per cent. variation of resistance or weight be allowed in all conductors.

9. Resolved, That an allowance of 1 per cent. increased resistance as calculated from the diameter be allowed on all tinned copper between Nos. 22 and 12 gauges inclusive.

Note to 1st and 3rd.

The figures inserted have been calculated for 60° F. from the figures 0·1469 per metre gramme for hard drawn and 0·1440 for annealed at 32° F. by Matthiessen's formula.

$$R_{t^\circ} = \frac{R_{32^\circ}}{1 - \cdot 00215006(t-32) + \cdot 00000279(t-32)^2}$$

A. H. Howard,
Honorary Secretary.

From the above data the following formulae are obtained:

Solid Wires.

Copper weighs 555 lb. per cubic foot at 60° F. Specific gravity = 8·912.

The resistance of annealed high conductivity commercial copper is:

Resistance per cubic inch =	0·0000066788 standard ohm.		
Resistance per cubic cm. =	0·00000169639	„	„
Resistance of 100 inches weighing 100 grains =	0·150151	„	„
Resistance per mile . . =	$\frac{0{\cdot}042317}{\text{area in square inches}}$.		

Resistance per yard . . = $\dfrac{0{\cdot}000024044}{\text{area in square inches}}$.

Resistance per mil-foot = 10·2044 standard ohms.

The resistance of hard-drawn high conductivity commercial copper is:

Resistance per cubic inch =	0·000000681327 standard ohm.
Resistance per cubic cm. =	0·00000173054 ,, ,,
Resistance of 100 inches weighing 100 grains . =	0·153181 ,, ,,

Resistance per mile . . = $\dfrac{0{\cdot}0431689}{\text{area in square inches}}$.

Resistance per yard . . = $\dfrac{0{\cdot}0000245277}{\text{area in square inches}}$.

Resistance per mil-foot = 10·4099 standard ohms.

Weight per mile . . . = 20350 × area in square inches.

Weight per yard . . . = 11·5625 × area in square inches.

Cables.

A lay of twenty times the pitch diameter is adopted as a standard, and the resistance in parallel of the wires is taken as the resistance of the cable.

Resistance of	3-strand cable =	0·33742	× resistance	of each	wire.
,,	4 ,,	0·253065		,,	,,
,,	7 ,,	0·1443557		,,	,,
,,	12 ,,	0·084355		,,	,,
,,	19 ,,	0·0532424		,,	,,
,,	37 ,,	0·0273493		,,	,,
,,	61 ,,	0·0165911		,,	,,
,,	91 ,,	0·0111222		,,	,,
Weight of	3-strand cable =	3·03678	× weight	of each	wire.
,,	4 ,,	4·04904		,,	,,
,,	7 ,,	7·07356		,,	,,
,,	12 ,,	12·1471		,,	,,
,,	19 ,,	19·2207		,,	,,
,,	37 ,,	37·4414		,,	,,
,,	61 ,,	61·7356		,,	,,
,,	91 ,,	92·0134		,,	,,

The above formulae give the standards, but a variation of 2 per cent. in resistance or weight is allowed for losses in manufacture.

These figures have been adopted by all of the parties represented and it is hoped that they may become the universal standard for Great Britain.

The measurements made by Dr. Matthiessen were for the purpose of determining the best metal to use for a standard resistance, and the permanence of the resistance was of more importance than the actual numerical value.

The specific gravity of the copper was not taken, and as the results are given by length and weight they afford no means of determining the resistance of a wire of any given diameter. In addition B.A. units have been confounded with standard ohms so that discrepancies have arisen from both causes.

The resistance of a stranded conductor varies according to the lay of the wires. Some makers use a lay of twelve times the pitch diameter, while others go so high as thirty times the pitch diameter; twenty was adopted as an average figure, and the resistances calculated from the actual length of the wires, viz. 1·01226 times the length of the cable for all except the centre wire.

As the Post Office specifications will be issued in accordance with the above report, and, as all the manufacturers mentioned will include the same figures in their catalogues, there seems little doubt that these standards will be adopted throughout Great Britain.

TABLE D. ELECTRIC

Gauge. S.W.G.	Diameter.		Effective Area.		Resistance at 60° F.
	Inches.	Millimetres.	Square Inch.	Circular Mils.	Ohms per 1000 Yards.
22	0·028	0·7112	0·0006158	784	39·05
21	0·032	0·8128	0·0008042	1024	29·90
3/25	0·043	1·092	0·0009311	1185	25·82
20	0·036	0·9144	0·001018	1296	23·62
19	0·040	1·016	0·001257	1600	19·13
3/23	0·052	1·321	0·001341	1707	17·94
18	0·048	1·219	0·001810	2304	13·28
3/22	0·060	1·524	0·001825	2323	13·18
7/25	0·060	1·524	0·002177	2771	11·05
17	0·056	1·422	0·002463	3136	9·762
3/20	0·078	1·981	0·003016	3839	7·972
16	0·064	1·626	0·003217	4096	7·478
7/23	0·072	1·829	0·003135	3991	7·670
15	0·072	1·829	0·004072	5184	5·904
7/22	0·084	2·134	0·004266	5431	5·636
14	0·080	2·032	0·005027	6400	4·784
7/21½	0·090	2·286	0·004896	6233	4·910
7/21	0·096	2·438	0·005571	7092	4·316
13	0·092	2·337	0·006648	8464	3·617
7/20	0·108	2·743	0·007052	8977	3·410
12	0·104	2·642	0·008495	10820	2·831
7/19	0·120	3·048	0·008708	11080	2·761
11	0·116	2·946	0·01057	13460	2·275
7/18	0·144	3·658	0·01254	15960	1·918
10	0·128	3·251	0·01287	16380	1·868
9	0·144	3·658	0·01629	20740	1·476
7/17	0·168	4·267	0·01706	21720	1·410
19/20	0·180	4·572	0·01942	24340	1·257
8	0·160	4·064	0·02011	25600	1·195
7/16	0·192	4·877	0·02227	28350	1·080
19/19	0·200	5·080	0·02360	30040	1·019
7/15	0·216	5·486	0·02822	35920	0·8523
19/18	0·240	6·096	0·03399	43270	0·7074
7/14	0·240	6·096	0·03483	44340	0·6903
19/17	0·280	7·112	0·04627	58900	0·5197
19/16	0·320	8·128	0·06039	76880	0·3981
19/15	0·360	9·144	0·07650	97380	0·3143
19/14	0·400	10·16	0·09442	120200	0·2547
37/16	0·448	11·38	0·1176	149700	0·2045
19/13	0·460	11·68	0·1249	159000	0·1926
37/15	0·504	12·80	0·1489	189550	0·1615
19/12	0·520	13·21	0·1595	203040	0·1507
37/14	0·560	14·22	0·1838	233980	0·1309
37/13	0·644	16·36	0·2431	309470	0·0989
37/12	0·728	18·49	0·3105	395270	0·0774
61/13	0·828	21·03	0·4008	510220	0·0600
61/12	0·936	23·77	0·5120	651780	0·0469
91/12	1·114	29·06	0·7638	857750	0·03148
91/11	1·276	32·41	0·9504	1209860	0·02530

LIGHT AND POWER CABLES.

Resistance at 60° F.		Weight.	Current.		
Ohms per Mile.	Ohms per Kilometre.	Pounds per Mile.	At 1000 Amps. per Square Inch.	I.E.E. Standard.	Kennelly's Rule.
68·72	42·70	12·52	0·6158	1·7	2·62
52·62	32·70	16 37	0 8042	2·2	3·2
45·45	28·24	19·42	0·9311	2·45	3·66
41·57	25·83	20·72	1 0179	2·6	3·83
33·67	20·92	25 58	1·2566	3·2	4·49
31·57	19·62	27·96	1·341	3·3	4·81
23·38	14 53	36·83	1·8096	4·2	5·90
23·19	14·41	38·05	1 825	4·26	6·06
19·44	12·08	45·23	2·177	4·92	6·06
17·18	10·68	50·12	2·4630	5·4	7·42
14·92	8·718	62·92	3·016	6·4	8·84
13·16	8·178	65·47	3·2170	6·8	9·06
13·50	8·389	65·12	3·135	6·64	9·90
10·39	6·456	82·87	4·0715	8·2	10 81
9·92	6·164	88·63	4·266	8·54	11·45
8·419	5·232	102·3	5·0265	9·8	12·67
8·643	5·371	101·8	4·896	9·56	12·72
7·596	4·720	115·8	5 571	10·63	14·00
6·366	3·956	135·3	6·6476	12·4	16·17
6·001	3·729	146·6	7·052	12·9	16·71
4·982	3·096	172·9	8·495	15·0	18·78
4·860	3·020	180·9	8·708	15·34	19·56
4·004	2·488	215·1	10 568	18·0	22·12
3·375	2·097	260·5	12·54	20·68	25 72
3·228	2·043	261·9	12·868	21·0	25·65
2·598	1·614	331·5	16·286	27·0	30·6
2·480	1·541	354·5	17·06	26·62	32·41
2 213	1·375	398·3	19·12	29·23	35·38
2·104	1 307	409·2	20·106	31·0	35·84
1·900	1·181	463·1	22·27	33·12	39·63
1·793	1·114	491·7	23·60	34·74	41·44
1·500	0·9321	586·2	28·22	40·22	47·26
1·245	0·7736	707·9	33·99	46·85	54·38
1·215	0·7550	723·6	34·83	47·80	55·34
0·9147	0·5684	963·3	46·27	60·33	68·65
0·7007	0 4354	1,258·0	60·39	75·06	84·07
0·5532	0·3437	1,593 0	76·50	91·12	100·08
0·4482	0·2785	1,966·0	94·42	108·3	117·21
0·3599	0·2236	2,451·0	117·6	129·6	138 56
0·3389	0·2106	2,601·0	124·9	136·2	144·55
0·2842	0·1766	3,103 0	148·9	157·3	165 34
0·2653	0·1649	3,323·0	159·5	166·4	173·73
0·2303	0·1431	3,830·0	183·8	187·0	193 65
0·1741	0·1082	5,066·0	243·1	235·2	238·82
0·1363	0·0847	6,474·0	310·5	287·4	287·03
0·1056	0·0656	8,353·0	400 8	354·3	347·71
0·0827	0·05136	10,674·0	512·0	433·1	417·92
0·0554	0·03443	15,925·0	763·8	600·1	564·75
0·0445	0·04453	19,811·0	950·4	719·3	665·26

TABLE E.

DIFFERENCES BETWEEN WIRE GAUGES.

No.	Brown & Sharpe's.	Old English or London.	Stubs' or Birmingham.	Wire Gauge Standard.
0000	·460	·454	·454	·400
000	·40964	·425	·425	·372
00	·36480	·380	·380	·348
0	·32495	·340	·340	·324
1	·28930	·300	·300	·300
2	·25763	·284	·284	·276
3	·22942	·259	·259	·252
4	·20431	·238	·238	·232
5	·18194	·220	·220	·212
6	·16202	·203	·203	·192
7	·14428	·180	·180	·176
8	·12849	·165	·165	·160
9	·11443	·148	·148	·144
10	·10189	·134	·134	·128
11	·09074	·120	·120	·116
12	·08081	·109	·109	·104
13	·07196	·095	·095	·092
14	·06408	·083	·083	·080
15	·05706	·072	·072	·072
16	·05082	·065	·065	·064
17	·04525	·058	·058	·056
18	·04030	·049	·049	048
19	·03589	·040	·042	·040
20	·03196	·035	·035	·036
21	·02846	·0315	·032	·032
22	·025347	·0295	·028	·028
23	·022571	·027	·025	·024
24	·0201	·025	·022	·022
25	·0179	·023	·020	·020
26	·01594	·0205	·018	·018
27	·014195	·01875	·016	·0164
28	·012641	·0165	·014	·0148
29	·011257	·0155	·013	·0136
30	·010025	·01375	·012	·0124
31	·008928	·01225	·010	·0116
32	·00795	·01125	·009	·0108
33	·00708	·01025	·008	·010
34	·0063	·0095	·007	·0092
35	00561	·009	·005	·0084
36	·005	·0075	·004	·0076
37	·00445	·0065	...	...
38	·003965	·00575	...	...
39	·003531	·005	...	...
40	·003144	·0045	...	...

TABLE F.

USEFUL CONSTANTS.

1 inch = 25·4 millimetres.
1 gallon = ·1605 cubic foot = 10 lb. of water at 62° F.
1 knot = 6080 feet per hour.
Weight of 1 lb. in London = 445,000 dynes.
1 pound avoirdupois = 7000 grains = 453·6 grammes.
1 cubic foot of water weighs 62·3 lb.
1 cubic foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.
1 cubic foot of hydrogen at 0° C. and 1 atmosphere, weighs ·00557 lb.
1 foot-pound = $1{\cdot}3562 \times 10^7$ ergs.
1 horse-power-hour = 33000 × 60 foot-pounds.
1 electrical unit = 1000 watt-hours.
Joule's equivalent to suit Regnault's H, is $\begin{cases} \text{774 ft.-lb.} = \text{1 Fah. unit.} \\ \text{1393 ft.-lb.} = \text{1 Cent. ,,} \end{cases}$
1 horse-power = 33000 foot-pounds per minute = 746 watts.
Volts × amperes = watts.
1 atmosphere = 14·7 lb. per square inch = 2116 lb. per square foot = 760 mm. of mercury = 10^6 dynes per sq. cm. nearly.
A column of water 2·3 feet high corresponds to a pressure of 1 lb. per sq. inch.
Absolute temp., $t = \theta°$ C. + 273°·7.
Regnault's H = 606·5 + ·305 $\Theta°$ C. = 1082 + ·305 $\Theta°$ F.

$$p\,u^{1{\cdot}0646} = 479.$$

$$\log_{10} p = 6{\cdot}1007 - \frac{B}{t} - \frac{C}{t^2},$$

where $\log_{10} B = 3{\cdot}1812$, $\log_{10} C = 5{\cdot}0871$,
p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

$$\pi = 3{\cdot}1416.$$

1 radian = 57·3 degrees.
To convert common into Napierian logarithms, multiply by 2·3026.
The base of the Napierian logarithms is $e = 2{\cdot}7183$.
The value of g at London = 32·182 feet per second per second.

TABLE G.

LOGARITHMS AND ANTI-LOGARITHMS.

Logarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

LOGARITHMS—*continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	5	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9939	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Antilogarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS—*continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	8	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5967	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8954	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE H.

NATURAL SINES AND TANGENTS.

Natural Sines.

—	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
°															
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10
45	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9

NATURAL SINES—*continued.*

—	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
°															
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	1·0000 nearly	1·0000 nearly	1·0000 nearly	1·0000 nearly	1·0000 nearly	0	0	0	0	0

Natural Tangents.

—	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
°															
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	14
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	17
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	10	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	10	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38

Natural Tangents—*continued.*

—	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
°															
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	23	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	$\bar{0}$057	$\bar{0}$145	$\bar{0}$233	$\bar{0}$323	$\bar{0}$413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1351	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	74	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	96
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	118
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2·9042	9208	9375	9544	9714	9887	$\bar{0}$061	$\bar{0}$237	$\bar{0}$415	$\bar{0}$595	29	58	87	115	144
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	82	122	162	203
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	94	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4·7046	7453	7867	8288	8716	9152	9594	$\bar{0}$045	$\bar{0}$504	$\bar{0}$970	73	146	219	292	365
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	175	262	350	437
80	5·6713	7297	7894	8502	9124	9758	$\bar{0}$405	$\bar{1}$066	$\bar{1}$742	$\bar{2}$432					
81	6·3138	3859	4596	5350	6122	6912	7920	8548	9395	$\bar{0}$264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}$285					
83	8·1443	2636	3863	5126	6427	7769	9152	$\bar{0}$579	$\bar{2}$052	$\bar{3}$572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10 58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·05	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24 90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44 07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114 6	142·2	191·0	286·5	573·0					

Angle.	Radians.	Sine.	Tangent.	Co-tangent.	Cosine		
0°	0	0	0	∞	1	1·5708	90°
1	·0175	·0175	·0175	57·2900	·9998	1·5533	89
2	·0349	·0349	·0349	28·6363	·9994	1·5359	88
3	·0524	·0523	·0524	19·0811	·9986	1·5184	87
4	·0698	·0698	·0699	14·3006	·9976	1·5010	86
5	·0873	·0872	·0875	11·4301	·9962	1·4835	85
6	·1047	·1045	·1051	9·5144	·9945	1·4661	84
7	·1222	·1219	·1228	8·1443	·9925	1·4483	83
8	·1396	·1392	·1405	7·1154	·9903	1·4312	82
9	·1571	·1564	·1584	6·3138	·9877	1·4137	81
10	·1745	·1736	·1763	5·6713	·9848	1·3963	80
11	·1920	·1908	·1944	5·1446	·9816	1·3788	79
12	·2094	·2079	·2126	4·7046	·9781	1·3614	78
13	·2269	·2250	·2309	4·3315	·9744	1·3439	77
14	·2443	·2419	·2493	4·0108	·9703	1·3265	76
15	·2618	·2588	·2679	3·7321	·9659	1·3090	75
16	·2793	·2756	·2867	3·4874	·9613	1·2915	74
17	·2967	·2924	·3057	3·2709	·9563	1·2741	73
18	·3142	·3090	·3249	3·0777	·9511	1·2566	72
19	·3316	·3256	·3443	2·9042	·9455	1·2392	71
20	·3491	·3420	·3640	2·7475	·9397	1·2217	70
21	·3665	·3584	·3839	2·6051	·9336	1·2043	69
22	·3840	·3746	·4040	2·4751	·9272	1·1868	68
23	·4014	·3907	·4245	2·3559	·9205	1·1694	67
24	·4189	·4067	·4452	2·2460	·9135	1·1519	66
25	·4363	·4226	·4663	2·1445	·9063	1·1345	65
26	·4538	·4384	·4877	2·0503	·8988	1·1170	64
27	·4712	·4540	·5095	1·9626	·8910	1·0996	63
28	·4887	·4695	·5317	1·8807	·8830	1·0821	62
29	·5061	·4848	·5543	1·8040	·8746	1·0647	61
30	·5236	·5000	·5774	1·7321	·8660	1·0472	60
31	·5411	·5150	·6009	1·6643	·8572	1·0297	59
32	·5585	·5299	·6249	1·6003	·8480	1·0123	58
33	·5760	·5446	·6494	1·5399	·8387	·9948	57
34	·5934	·5592	·6745	1·4826	·8290	·9774	56
35	·6109	·5736	·7002	1·4281	·8192	·9599	55
36	·6283	·5878	·7265	1·3764	·8090	·9425	54
37	·6458	·6018	·7536	1·3270	·7986	·9250	53
38	·6632	·6157	·7813	1·2799	·7880	·9076	52
39	·6807	·6293	·8098	1·2349	·7771	·8901	51
40	·6981	·6428	·8391	1·1918	·7660	·8727	50
41	·7156	·6561	·8693	1·1504	·7547	·8552	49
42	·7330	·6691	·9004	1·1106	·7431	·8378	48
43	·7505	·6820	·9325	1·0724	·7314	·8203	47
44	·7679	·6947	·9657	1·0355	·7193	·8029	46
45	·7854	·7071	1·0000	1·0000	·7071	·7854	45
		Cosine	Co-tangent	Tangent	Sine	Radians	Angle

ANSWERS

EXERCISES I A.

(1) 60000. **(2)** 0·75. **(3)** 25. **(4)** 40. **(5)** 25920. **(6)** 252. **(7)** 17·7174. **(8)** 1800. **(9)** 15. **(10)** 12·8. **(11)** 112·7. **(12)** 1 min. **(13)** 1·18. **(14)** 2·934 oz. **(15)** 1·25775 grms. silver, 0·011682 grm. H. **(16)** 3·4 : 1. **(17)** 2·4021. **(20)** 88·2 and 117·6. **(21)** 64·5 and 15·5. **(22)** 7·68.

EXERCISES I B.

(1) 0·4. **(2)** 0·08. **(3)** 45 ft. **(4)** 20 ft. **(5)** 5 ohms. **(6)** 31·5. **(7)** 0·0412. **(8)** double it on itself. **(9)** 3·29. **(10)** 0·1018. **(11)** 2·07. **(12)** 12·2. **(13)** 1·732 : 1·414. **(14)** 24 : 25. **(15)** 0·0508. **(16)** 0·000072. **(17)** 15625. **(18)** 9 : 1 and 1 : 39·37. **(19)** $126\frac{9}{16}$. **(20)** 0·587 inch. **(21)** 22·4 microhm. **(22)** 0·592 microhm. **(23)** 0·636 microhm. **(24)** $20\frac{5}{6}$ mils. **(25)** 25 metres and $56\frac{1}{4}$ metres. **(26)** 0·003 sq. inch. **(27)** 50 mils. **(28)** 2·44 : 1. **(29)** 0·2774 cm. **(30)** $l_i = \cdot 688\ l_c$. **(31)** 43·2. **(32)** 9 mils. **(33)** 3·29. **(34)** 98·5 per cent. **(35)** 2. **(36)** 110·25. **(37)** 1 : 100. **(38)** 1 : 100. **(39)** 1 : 16. **(40)** 0·000009. **(41)** 1·7 microhms. **(42)** 120, 140, and 160. **(43)** 500 grms. **(44)** 2·74 lbs. **(45)** ·000000312. **(46)** 0·9025. **(47)** 95 per cent. **(48)** 13·474 ; 26·949 ; 25·932. **(49)** 1 : 2·408 and 1 : 1·304. **(50)** 8383 cms. and 4·1 mm. **(51)** $\frac{78{\cdot}54\ R}{l\,m^2}$ microhms and $\frac{\cdot 7854\ lg}{m^2}$ grms. **(52)** $R_t = r\,(1+a t)$ where $a = \frac{R-r}{n}$. **(53)** 24·54. **(54)** 1·647 microhms. **(55)** 9°·6 and 20°·5. **(56)** 36°·5. **(57)** 1·6192. **(58)** 22·9 and 17·3. **(59)** 1282·1. **(60)** 120·7245. **(61)** 103·6 ; 459·5 ; 722·2 cms. **(62)** 32·1 mils. **(63)** 0·0883. **(64)** 81 and 919.

EXERCISES I C.

(1) 1·46. **(2)** 110. **(3)** $2\frac{1}{2}$. **(4)** 4 amps. in each case. **(5)** ·01. **(6)** 25·25 m. amp. **(7)** 104 ohms. **(8)** 1·04 amp. **(9)** 0·2 amp. **(10)** 126 m. amp. **(11)** 50 %. **(12)** 165 ohms. **(13)** $\frac{20}{33}$ amp. **(14)** 1·9 volts. **(15)** 8 ohms. **(16)** 161·7 volts. **(17)** $16\frac{2}{3}$ ohms. **(18)** 0·1 ohm. **(19)** $n = \cdot 002 \times c\,(l\,r_m + r_b)$. **(20)** 77. **(21)** 0·2 amp. **(22)** 63. **(23)** 1. **(24)** 200 and 650 volts. **(25)** 2·2 volts. **(26)** 8 amps. **(27)** 1·3 ohms. **(28)** 12 volts.

EXERCISES I D.

(1) 2970000. **(2)** 2640000 foot-pounds. **(3)** 17820000 joules. **(4)** 3300 watts. **(5)** 10000 watts. **(6)** 66 : 100. **(7)** 33 : 40. **(8)** 100·5 H.P. **(9)** 9·6. **(10)** 66·3. **(11)** 4·8. **(12)** 81. **(13)** 0·134 H.P. **(14)** 6 volts. **(15)** 99·5 volts ; $132\frac{2}{3}$ ohms. **(16)** 3·4 ohm. **(17)** 10 amps. **(18)** 285 lamps and 145 amps. **(19)** 120 amps. ; 241·6 H.P. **(20)** 1 : 200. **(21)** 1 : 16. **(22)** 4 times. **(23)** (a) $\frac{r_1 r_2}{(r_1+r_2)^2}$; (b) $\frac{(r_1+r_2)^2}{r_1 r_2}$. **(24)** 72·3. **(25)** $1{\cdot}356 \times 10^7$. **(26)** ·7372. **(27)** 238. **(31)** 36×10^{12}. **(34)** $4{\cdot}2 \times 10^7$. **(35)** 1·12. **(36)** $1333\frac{1}{3}$ watts ; 59000 foot-pounds. **(37)** 27·12. **(38)** 746. **(39)** 254880000 foot-pounds.

(40) 7374 foot-pounds and 10^{11} ergs. **(42)** 1*s.* $3\frac{3}{4}$*d.* **(43)** 75 units; £1 5*s.* **(44)** $3\frac{1}{7}$ pence. **(45)** input 255 watt-hours and 677025 foot-pounds; output 224·1 watt-hours and 594985·5 foot-pounds.

EXERCISES II A.

(1) 8 volts. **(2)** 22·5. **(3)** 115. **(4)** $3\frac{9}{22}$. **(5)** ·0244. **(6)** ·00008. **(7)** $2\frac{5}{11}$ volts and 107·1 amps. **(8)** 10 yards. **(9)** 6 volts 1·2 amps. **(10)** 0·4 and 2. **(11)** 1 ohm and 17 volts. **(12)** 15 and 6. **(13)** $1\dot{3}$ ohms; 487·2 volts; 48 volts. **(14)** ·4 amp.; 1·4, 2·4, 4·6 volts; 8·4 volts. **(15)** $53\frac{1}{33}$ amps.; 1·9 ohms. **(16)** 30 volts. **(17)** 5 volts; 1 amp.; 0·8 amp. **(18)** 2. **(19)** 25 and 5 ohms. **(20)** 90 ohms. **(21)** 2·5 volts. **(22)** 8 and 2 ohms. **(23)** 1·65 ohms; $41\frac{1}{4}$ volts. **(24)** ·07128 sq. inch. **(25)** $8\frac{8}{11}$ %. **(26)** $61\frac{1}{9}$ %. **(27)** $\frac{2}{5}$ of A D C. **(28)** ·001625 sq. inch. **(29)** 1·8. **(30)** 406·2 feet. **(31)** 5 amps.; 31·25 volts; 100 volts. **(32)** 1·1 ohms.

EXERCISES II B.

(1) $142\frac{6}{7}$. **(2)** 1·38. **(3)** 2. **(4)** R. **(5)** 1. **(6)** 5·89. **(7)** 3·29; 1·75. **(8)** ·04. **(9)** 2·98 ohms. **(10)** 4. **(11)** 4. **(12)** 1·2. **(13)** 4 or $\frac{1}{4}$. **(14)** $33\frac{1}{3}$. **(15)** 20 : 1. **(16)** $\sqrt{2} : 1$. **(17)** $l_1 s : l s_1$. **(18)** 1·2. **(19)** ·375 and 1·125. **(20)** $\frac{16}{127}$; $\frac{12}{127}$. **(22)** $\frac{10}{19}$, $\frac{5}{19}$, $\frac{4}{19}$. **(23)** 16; 8; 4. **(24)** 20 : 15 : 24 and 11·8 amps. **(25)** 10. **(26)** 1·46. **(27)** 28. **(28)** 200 volts. **(29)** ·2. **(30)** −·25; 1·45; ·95; ·25. **(31)** 1·67 and 0·83. **(32)** 6 and 30. **(33)** $r_2 = 100$; $r_3 = 50$; $r_4 = 25$. **(34)** $263\frac{3}{119}$; 250. **(35)** 30. **(36)** 3 : 2. **(37)** 4 : 1. **(38)** 10000. **(39)** $416\frac{2}{3}$. **(40)** 15·7; 21·99; 42·29; and 0·5496 volt. **(41)** ·4983; ·49667; ·495; ·4934; ·4918; ·49.

EXERCISES II C.

(1) 17500. **(2)** $533333\frac{1}{3}$. **(3)** ·0000738 amp. **(4)** 1500. **(5)** 2·4 miles. **(6)** $33\frac{1}{3}$ mls. from beginning and 800 ohms. **(7)** $\frac{7}{10}$. **(8)** 200 ohms. **(9)** $428\frac{4}{7}$. **(10)** 15·51 m. amp. **(11)** 127·66 megohm. **(13)** 1·45 : 1. **(14)** $15{\cdot}286 \times 10^3$ and 29·1 megohms.

EXERCISES II D.

(1) ·043 mf. **(2)** 103·1 sq. cms. **(3)** 222 units. **(4)** 20 : 27. **(5)** ·14 mf. **(6)** 3 in parallel and one in series. **(7)** 2 : 1. **(8)** ·314 mf.

EXERCISES II E.

(1) ·164; ·267; ·186. **(2)** 25 cells. **(3)** ·2 amp. **(4)** $1\frac{1}{3}$ amps. **(5)** 1·3 ohms current increased as 1·3 : 1·22. **(6)** in no other way. **(7)** 4 amps. **(8)** 63 cells. **(9)** 8 cells in a row; 1 amp. **(10)** one cell in opposition. **(11)** same strength. **(12)** 2 rows; ·008 amp. **(13)** 60 %. **(14)** ·36 amp. in both cases; 20 % and 80 %. **(15)** $35\frac{5}{7}$. **(16)** 6. **(17)** 16 cells.

EXERCISES II F.

(1) 90000; 1296000. **(2)** 2200 watts or 528 calories per sec. **(3)** 7 ohms. **(4)** 1 : 16. **(5)** 10°·8. **(6)** ·4016 grm. **(7)** 150 : 100. **(8)** 1 : 3. **(9)** 518400. **(10)** 13·536 calories. **(11)** $33\frac{1}{3}$ secs. **(12)** 11·9 secs. **(13)** [b]$R_2 : R_1$. **(16)** ·02 calorie. **(17)** ·2478 pence. **(18)** $C_1 : C_2 = d_1^{\frac{3}{2}} : d_2^{\frac{3}{2}}$; 1 amp. **(19)** about $\frac{1}{2}$° C. increase. **(20)** ·1078 cm. **(21)** 203·5. **(22)** 2600. **(23)** approx. equal. **(24)** 450 amps.

EXERCISES II G.

(1) ·00107328. **(2)** ·186912 grm. H. and 1·49148 grms. O. **(3)** ·67086 grm. **(4)** ·0108 amp. **(5)** 12 min. 42 sec. **(6)** ·000326 and 31·4. **(7)** ·674 grm. **(8)** copper 6560·6 grms., mercury 20747·23 grms., copper 3280·3 grms., H. 103·84 grms., caustic soda 4153·5 grms. **(9)** caustic soda 1·4976 grms., lead 3·87504 grms., silver 4·04352 grms., mercury 7·488 grms. **(10)** ·00479 mm. **(11)** 10333 H.; 5166 O. **(12)** ·10836 grm. **(13)** ·1733 c.cm.

EXERCISES III.

(1) 118. **(2)** 200 volts. **(3)** 3·45 H.P. **(4)** 240. **(5)** 59. **(6)** 108. **(7)** ·06534 sq. inch. **(8)** ·07841 sq. inch. **(9)** ·09557 sq. inch. **(10)** 69·23 amps. **(11)** 120 %. **(13)** 150 amps.; 3·75 volts; 86·7 %. **(14)** ·0885 inch. **(15)** 78 and 78. **(16)** 2043 amps. per sq. inch. **(17)** ·25 sq. inch. **(18)** 2000 ft.; 206·04 volts; 3·02 %. **(19)** 5 : 4, 5 : 2, 5 : 8, 5 : 16. **(20)** ·103 sq. inch; ·0104 sq. inch; ·0012 sq. inch. **(21)** ·0776 sq. inch; ·0814 sq. inch; ·0852 sq. inch; ·089 sq. inch. **(23)** $\frac{2}{3}$ ohm; $\frac{1}{5}$ sq. inch; 5 volts; 600 amps. per sq. inch. **(24)** ·436 sq. inch. **(25)** i, 0 and 3·2; ii, 1·6 and 1·6; $\frac{5}{6}$ sq. inch and $\frac{5}{12}$ sq. inch; 203·2 volts and 201·6 volts. **(26)** ·058 sq. inch. **(27)** 1·3 inch. and ·449 inch. **(28)** 1·6 sq. inch.; ·4 sq. inch. **(29)** (*a*) 4·02 sq. inch and 5145·6 lb.; (*b*) 1·005 sq. inch. and 2316·8 lb. **(30)** 577·3 yds. **(31)** ·168 sq. inch and 1028 lb. **(32)** 74·6 amps.; ·56 ohm. **(33)** 1·45 inch.; ·607 inch. **(34)** i, reduced 75 %; ii, to $\frac{1}{4}$; iii, 4 times. **(35)** $V_{ab} = 224{\cdot}242$ volts; $V_{bc} = 209{\cdot}815$ volts. **(36)** 500 amps. per sq. inch. **(37)** ·8 sq. inch; 22·5 volts per mile; 1011·25 volts. **(38)** 4500 watts; £65·7; £65·52. **(39)** ·9 sq. inch; 4000 watts; £58·4; £73·71. **(43)** ·0457 sq. inch.

EXERCISES IV A.

(1) 147·15 dynes. **(2)** 3 : 5 and 3 : 5. **(3)** 6. **(4)** 15. **(5)** 24 and 12. **(6)** 15. **(7)** 261·6. **(8)** 109. **(10)** 10. **(11)** 2 : 3. **(12)** increased 2·14 times. **(13)** 4 : 1. **(14)** 90 lb. **(15)** 1·5625. **(16)** ·336. **(17)** 5. **(18)** 2·43. **(19)** ·283. **(20)** 3·17. **(21)** ·05. **(22)** ·406. **(23)** 1·27. **(24)** 64 : 125. **(25)** 2·83 cm. from the weaker. **(26)** 7·07 dynes. **(27)** 41·68. **(28)** unity. **(29)** 30°. **(30)** 1 dyne; 1 line; 400 π; 4 π; 400 π.

EXERCISES IV B.

(1) 3 : 2. **(2)** 4 : 3. **(3)** 9 dyne cms. **(4)** 18 dyne cms. **(5)** 2·29 dynes. **(6)** 10 : 9. **(7)** 1 : 1. **(8)** ·18; ·192; ·208. **(9)** 50. **(10)** 555·5. **(11)** 50. **(12)** 93·25 dyne cms. **(13)** 3 : 1. **(14)** 26°. **(16)** 46°. **(17)** F. **(18)** 3 F.

EXERCISES IV C.

(1) 2·5 dynes. **(2)** 5°. **(3)** 3·67. **(4)** 760°; 1680°. **(5)** 230°. **(6)** 293·3°. **(7)** 7 : 11. **(8)** 257; 320. **(9)** 750°. **(10)** 17 : 27. **(11)** 150°. **(12)** 17 : 24. **(13)** 230°.

EXERCISES IV D.

(1) ·163. **(2)** ·225; 1·25. **(3)** 3 : 7. **(4)** 9 : 16. **(5)** $\frac{1}{47{\cdot}5}$. **(7)** 1 : 1·21. **(8)** 64 : 81 : 36. **(9)** 15 : 49. **(10)** 25·6 mins. **(11)** 128. **(12)** 545 : 7632 and 109 : 272.

EXERCISES V A.

(1) 1; 100; 10000 dynes and $\frac{1}{981}$; $\frac{100}{981}$; $10\frac{190}{981}$ grms.-weight. **(2)** (*a*) 12570; (*b*) 125·7 dynes; (*c*) 502·8 lines. **(3)** 1·257. **(4)** 3·9 amps. **(7)** 100·56 and 99. **(8)** ·8. **(9)** 494; 398·6. **(10)** 14·32. **(11)** 330 metres and 12·57. **(12)** 50 and 319·2. **(13)** 5 amps. **(14)** 562·5 C.G.S. units. **(15)** 67·5 and 4243.

EXERCISES V B.

(1) ·075. **(2)** ·0111; 908. **(3)** 4·16. **(4)** $333\frac{1}{3}$. **(5)** $\frac{1}{4200}$; 238·1. **(6)** 14520. **(7)** 4·61 amps. **(8)** 795. **(9)** 1000. **(10)** $\frac{1}{150}$; ·01; ·016 and 2·5; 7·95; 15·9 amps. **(11)** 4210. **(12)** 27900. **(13)** ·00973. **(14)** ·001; ·00125; ·0025. **(15)** 160; 400; 1120. **(16)** 3750. **(17)** 3520; 280. **(18)** 201·8. **(19)** 12716·4. **(20)** 2515. **(21)** 55609. **(22)** 37·5; 187·5; 75. **(23)** 7·6; 6. **(24)** 562·5 C.G.S. units.

EXERCISES V C.

(1) 6787·8 units; ·00404; 96000; 1680000. **(2)** 11908; 208390. **(3)** ·032576. **(4)** 43074. **(5)** ·002355. **(6)** 20392782. **(7)** 1621. **(8)** increased 3·08 times. **(9)** 844. **(10)** 23446. **(11)** 4732 and 7680; μ for cast-iron being 800 and 350. **(12)** 1232·25. **(13)** 2001530. **(14)** 14744.

EXERCISES VI A.

(1) 245·44 ft. **(2)** 1·1781 cu. inch. **(3)** 2539. **(4)** 266 ft. **(5)** 87·78 ohms. **(6)** ·423 inch. **(7)** 3·07 inch. **(8)** 14·6 mils. **(10)** 1·6875 inch. **(11)** 1·43 inch. **(14)** 1 : 16. **(15)** 2 : 1 and 1 : 4. **(16)** 40 mils. **(17)** 31·1 ohms.

EXERCISES VI B.

(1) 985 turns of wire, 13 mils diam. **(2)** 39·6 mils diam.; 51 ohms; 31·4 lb. **(3)** 76 mils diam.; 9·9 ohms; $83\frac{1}{4}$ lb. **(4)** ·0431 inch. **(5)** ·0456 inch. **(6)** 58 mils; 41 mils; 39·9 mils; 32·2 lb. **(7)** 32·7 mils. **(8)** 76·9 units. **(9)** 62·3 mils diam. and 924·4 yards.

EXERCISES VI C.

(2) 17·5 lb. **(3)** 89800; 223·6 lb. **(4)** 4430 lb. **(5)** 16130 and 104000.

EXERCISES VII A.

(1) ·25 volt. **(2)** ·047 volt. **(3)** ·006 volt. **(4)** 110 volts. **(5)** 8363636·36 C.G.S. lines. **(6)** ·48 volt. **(7)** 145·3 sq. cms. and 193·7 sq. cms. **(8)** 120 volts. **(9)** 30 r.p.m. increase. **(10)** 1149 r.p.m. **(11)** 7051282. **(12)** 10526·3. **(13)** 176·25 volts. **(14)** 12400. **(15)** ·01785 ohm. **(16)** 305·8 sq. cms. and 4640 turns. **(17)** 21344 ampere-turns. **(19)** i, 8363636·36; ii, 6610; iii, 5936.

EXERCISES VII B.

(1) 2038·74 grms. or $4\frac{1}{2}$ lb. **(3)** ·000531. **(4)** 2000 lb. **(5)** 34·25 lb. **(6)** ·983 lb.

EXERCISES VIII A.

(1) 9·125 Kw. and 80·4 %. **(2)** 16·6 amps. **(3)** 1875 watts; 1350 volts; 88·8 %. **(4)** 2400 watts; 204 volts; 88·3 %; 75·4 %. **(5)** 836 volts and 29·4 H.P. **(6)** 95·8 % and 39·9 ohms. **(7)** 200 watts; $444\frac{4}{9}$ watts. **(8)** 195 volts; 15 volts; 2·7 Kw. **(9)** 1000 watts. **(10)** 1429·5 volts and ·19125.

EXERCISES VIII B.

(**1**) 11520 watts ; 94·4 % ; 120 volts ; 122·49 volts. (**2**) 8800 watts ; 312·589 watts. (**3**) 104·96 volts and 124 amps. (**4**) 116·86 volts. (**5**) ·0193 ohm. ; 40·6 ohms. (**6**) 3·014 ; 302·5 ; 93·303 H.P. (**7**) 10780·5 watts ; 83·22 %. (**8**) 112 volts ; 163·5 amps. ; 114·1692 volts ; 160 amps. (**9**) 205 amps. ; ·0131 ohm ; 2·675 %. (**10**) 220 volts ; 104 amps. ; ·01 ohm. (**11**) 6·21 amps. ; 326·21 amps. ; 3·26 volts ; 108·26 volts ; 1063·4 watts ; 652 watts ; 33600 watts ; 35315 watts.

EXERCISES VIII C.

(**1**) ·625 ohm ; 122·625 amps. ; 109·905 volts ; 66·78 %. (**2**) 539 watts ; 275·6 watts ; 3600 watts ; 9000 watts. (**3**) ·5225 ohm ; 17·73 ohms ; 205·92 amps. ; 113·73 ; 89·6 %. (**4**) ·061782 ; 30·96 ohms ; 93·49 amps. ; 108·18 volts. (**5**) 9720 watts ; 540 watts ; 378 watts ; 162 watts.

EXERCISES IX A.

(**1**) $63\frac{1}{3}$ H.P. (**2**) 33 amps. (**3**) 95%. (**4**) 99 and 100·8 volts. (**5**) 74·6 % ; 79·6 %. (**6**) ·4896 ohm and 1524 watts. (**7**) 32 amps. ; 35 amps. (**8**) 250 amps. ; 28·68 H.P. (**9**) 98 volts ; 500 watts. (**10**) 91·6 % and 21 amps. (**11**) 96·135 volts. (**12**) $C_2 : C_1 = \cdot 8 : 1$ or $= \cdot 6 : 1$. (**14**) $\frac{n}{9{\cdot}55}$ or $\cdot 1047\, n$ radians. (**15**) 19·44 pounds-weight. (**16**) 14·58 pounds-weight. (**17**) 11·43 H.P. (**18**) 100 pound-feet ; 50 pounds-weight. (**19**) 437·5 pounds-weight. (**20**) 150 volts. (**21**) 480 volts ; 480 r.p.m. ; 1466·8. (**22**) 329 r.p.m. ; 80·2 H.P. ; 1268 lbs.

EXERCISES IX B.

(**1**) 7·54 miles per hr. (**2**) 16 amps. (**3**) 10·72 miles per hr. ; 12·35 miles per hr. (**4**) 22·05 H.P. ; 54·83 amps. (**5**) 52·4 amps. ; 5·59 H.P. (**7**) 103·4 pounds per ton. (**8**) 34·2 sec. and 251 feet. (**9**) 19·6 sec. and 143·7 feet. (**10**) 120·3 pounds-weight. (**11**) 2305·1 inch-pounds. (**12**) 63179·5 poundals ; 74822·3 foot-pounds ; 38·1 feet. (**13**) 7·92 miles per hour.

EXERCISES X.

(**1**) 9·2 H.P. and 84·6 %. (**2**) 99·5 volts ; 94 % ; 80·2 %. (**3**) 92·3 % ; 92·8 %. (**4**) 1002 r.p.m. ; 8·2 H.P.

EXERCISES XI.

(**2**) 17089 joules. (**3**) 51 cms. (**4**) 1788·16 cms. per sec. (**5**) $3333\frac{1}{3}$ dynes. (**7**) 24 pounds-weight. (**8**) 1736 foot-pounds. (**9**) 18. (**10**) 1 : 1 ; 1 : 3 ; 1 : 9. (**12**) 2000. (**13**) 10^{-4} ; 10^{-13} ; 10^{-9}. (**14**) 31·76. (**15**) 3·886. (**16**) ·4754. (**17**) 6·643. (**18**) 300. (**19**) $301155\frac{5}{9}$ foot-pounds ; 5019·26 pounds-weight ; 7·17 ; 4·09 sec.

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